Mathematics *and* Religion

OUR LANGUAGES OF SIGN AND SYMBOL

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Preface

At any of the world's great tourist sites—Paris, Tokyo, or Mexico City, for example—we typically see travelers using dictionaries to translate their native tongues into the local language, reminding us that our natural languages still divide us. At the same time, a language that seems to unite us wherever we go is the language of mathematics. Whether we are traveling in Germany, Indonesia, or Brazil, we can do business together because we agree that 2 + 2 = 4. The mathematical laws of gravity and the three dimensions of space allow us to find directions to New York's Central Park or fly home, over a curved planet, on a jetliner.

This book is about our languages, but with a focus on the privileged role that mathematics has in our ability to communicate about the world around us. In this sense, mathematics is our public language, but it is more than that also. As I hope you will see, mathematics leads us through science and brings us to questions about a greater reality called metaphysical reality, which we usually approach in the context of philosophy and religion.

For many years, my friends, students, and acquaintances have asked me how I—a university professor of mathematics, logic, and computer science—reconcile my profession with my other vocation as a Jesuit priest. How does a person reconcile science and religion, both intellectually and on a personal basis? This book on mathematics and religion has also allowed me to explore that question further.

Mathematics is a difficult language for the uninitiated, especially when we reach beyond elementary arithmetic, geometry, and algebra to what I call the modern formal language of mathematics—the language we apply, for example, to computer science. In this advanced realm of math, we also probe the sufficiency of mathematical systems themselves. Do these systems have the power to explain an ultimate reality, or are they more like tools we invent to simply do the job: to measure the construction of a bridge, send a rocket to the moon, or run a computer calculation? By pursuing these advanced areas of mathematics, we finally arrive at questions about the truth and consistency of any system, whether a scientific system that talks about nature or a theological system that talks about God.

I have organized this book in the following way. The first chapters focus on defining and explaining the three basic languages that concern us. Each has a particular kind of perception of reality and then a system of signs or symbols (a language) to convey that perception.

The first kind of perception is found in logic and mathematics, a purely mental kind of perception. It uses the language of formal signs. This language often seems to be an array of impenetrable hieroglyphics, such as the notation $\Sigma ar = (0, 1, +, \times, <)$. But, as we will see, everyone can understand this language to some degree. Of course, the language is also very specialized, the lingua franca of logicians and mathematicians.

The second kind of perception of the world is through empirical science, which uses representational language to convey that perception. This language speaks of physical realities such as weight, force, or mass, and is the language of physics, chemistry, geology, and neuroscience. Famously, for example, Newton gave us a representational language to talk about the force of gravity based on the mass and distance between objects such as planets. Einstein did likewise with a language that says energy equals mass in the particular arrangement of $E = mc^2$.

Finally, a third kind of human perception and language is metaphysical. It is a logical language just like mathematics, but it uses symbols (not signs) to explain perceptions of relationships, causes, and ultimate reality. These symbols have included the idea of a transcendent God, a being that surpasses all other beings, or a being that is in relation to the world, but is nevertheless beyond the world. Metaphysical language uses symbols to speak about the individuality and unity of things, the nature of the infinite, the scope of the universe, or the relationships we call community.

It took most of human history for us to arrive at our modern understanding of mathematical language. The largest section of this book is a survey of the evolution of our mathematical systems, which I find the best way to introduce mathematics to a general audience. In hindsight, we can now see the turning points in our growing knowledge of mathematics and reflect on the colorful stories of the people who moved the science of mathematics up to its present state. As we will see, the formalization of mathematical systems characterizes that current state.

After the history, we look at two of the most fundamental mathematical languages: propositional logic and first-order predicate logic. Please don't let these names—which we technically call L0 and L1—make the section on propositional logic (chapter 7) off-putting. This is fairly technical material, but I have tried to explain it in a narrative of natural language and have added accompanying appendices for students who wish to pursue this topic more extensively. For a general reader, this introduction to the formal language of logic offers a useful overview and a glimpse at how mathematicians and computer experts think and talk today.

The book concludes with a look at how we derive meaning from the semantics of our various languages—mathematical, empirical, and religious—and in what ways metaphysical questions have become more important as our culture grows more scientific. Our ability to understand our various kinds of languages helps us not only to take advantage of a complex scientific world but also to deepen our search for personal answers to the great metaphysical questions.

Personal experience often motivates the desire to embark on such metaphysical search. The crucial experience in my life came in my homeland of Spain, where I began as a philosophy student. I later pursued advanced studies in mathematics, attracted by its beauty, clarity, and technical power. I had been studying mathematics at the university for three years when, in May 1968, the student revolution spread the euphoria of cultural change across Western Europe. Traditional values such as religion, patriotism, and respect for authority were called into question. Equality, sexual liberation, and human rights were affirmed. This was also the decade of the Second Vatican Council (1962–65). Vatican II represented an effort to open the Catholic Church to the current culture on several fronts. The outcomes of this effort included: updating liturgical language, incorporating human rights values into the life of the church, recognizing the basic equality of all baptized, and acknowledging the right to religious freedom and the need to cooperate with other religions.

My own questioning in this period led me to study theology, beginning in the early 1970s in Frankfurt, Germany. In my own country, where I returned to teach, the long rule of General Franco was coming to an end with his death in 1975. The political transition to democracy was a major cultural change in Spain. Having entered the priesthood in these years, I also continued as a professional mathematician. Today I teach logic and mathematics in the department of computing at the Complutense University of Madrid, one of the main public universities in Spain.

In the past three decades, the world has seemed to change more rapidly than ever. Math and science have grown in importance. To the surprise of many, we also see a perennial return to religion. In such times, I believe that mathematics retains a privileged position because of its unique role in linking—by the principles of logic—science with philosophy and theology. In making this case, I speak as a Christian who values the interfaith spirit of our age and the age-old tradition of humanist learning.

While language is the means by which we convey meaning, I believe there is too little reflection on how the language of mathematics and the language of religion may share common characteristics. Stating that they are two alien types of language is too simple. Therefore, this book offers models for how the languages of science and religion complement each other. Science and religion live in a complex relationship (what in the last chapter I call Non-Symmetrical Magisteria). But they can both offer valid truths based on a common criterion of internal consistency and usefulness in the world.

I hope that this little book conveys to those curious about mathematics the rich world I have found in this discipline, and why I believe that seeing mathematics in new ways can increase our sense of the beauty of the world and our ability to find harmony between science and our faith traditions.

CHAPTER 1

Mathematics and Natural Sciences

Since the rise of modern science in the sixteenth century, mathematics has often been characterized as the language of nature. We often forget, however, that we have never stopped debating whether we can talk most accurately about the world by using only numbers or by also using physical models. Are the solar system and the movements in the night sky, for example, best understood by a series of numbers written on a sheet of paper, or when we view a wooden model of the planets and the solar system, so to speak, as the early scientists of the Renaissance did?

Although mathematics and natural science are closely bound together, they represent essentially two different kinds of language. Mathematics refers primarily to objects of the mind. Natural science refers to objects of our sense experience. In mathematics we use abstract formal signs (that is, the language of precise mental meaning and a language that we can manipulate mechanically). In contrast, natural science uses what we may call representational language that speaks of the physical objects which physics, chemistry, geology, and neuroscience study.

We can go deeper as well. At the heart of both mathematics and natural science lies the primary level of logic. Once we have logic, we are able to move on to mathematics and to natural science. At each of these levels, we perceive reality and then we use a type of language to express that perception.

Formal Signs in Logic and Mathematics

What we perceive at the level of logic is correct reasoning, an inference that one thing naturally leads to another. We can test such logical inference in formal models of logic or mechanically, as in a computer. But many times we perceive something as logical simply by the power of intuition: it immediately seems to be so. These are logical intuitions. They intuit that something is following the rules of logic. For example, "It is impossible that something be true and false at the same time" is a logical principle that we intuit is always valid. We call this the principle of noncontradiction.

The logic we intuit can also be put into a formal language. As evidenced by the abstract signs often seen in logic or mathematics, formal language consists of a finite series of signs that follow rules of syntax. The signs have no definite meaning until they are related to each other by these rules, and then we can interpret these strings of signs as true or false. That language of logic sets the stage for the language of mathematics.

The way to understand the relationship of logic and mathematics is to say that while mathematics includes logic, it cannot be reduced to formal logic. Mathematics has something more, a kind of mathematical intuition and freedom based on logic. In fact, if we reduce mathematics to pure formal logic, we end up with paradoxes, which amount to contradictions. The great mathematical ambition of the German Gottlob Frege (d. 1925) and the Englishman Bertrand Russell (d. 1970), both of whom wanted to reduce mathematics to formal logic, illustrated this paradox. The result, however—which they conceded—is that such an effort ends in paradoxes. So again, logic and mathematics are different despite many similarities.

Like logic, however, mathematics also begins with intuitive perceptions. Mathematics begins as a purely intellectual, intuition-driven exercise. One of the first great mathematicians, Euclid, proposed many of these natural intuitions. For example, the first Euclidean postulate expresses the mathematical intuition that between any two points a straight line segment can always be drawn. In applying mathematics, we give these intuitions another name: mathematical axioms, which amount to beliefs that we presume to be true. (Hereafter, we use the terms "axiom" and "postulate" interchangeably since they have the same meaning.)

So mathematics is built of two parts, the axioms and the mathematical statements that seem logical. However, since axioms are basic intuitions, and they are the foundation of a particular mathematical system, axioms are not valid in all systems. What remains valid in all systems is the logic of mathematical propositions. As we see later in the book, this realization has created a variety—or pluralism—of mathematical systems. Yet in each one, certain logical propositions must always be valid. We can turn again to Euclid to illustrate this point. The axioms that Euclid began with ensured that his geometry was consistent and logical. However, not all forms of mathematics begin with Euclid's axiom. Basic arithmetic does not use those axioms, and thanks to modern revolutions in math, today we have non-Euclidean geometry, which uses axioms different from Euclid's.

As we can see, if axioms and logical principles are mixed in the wrong ways we end up with paradoxes, which means that we can deduce a proposition, but also its negation. It would seem that paradoxes would always be a bad thing, since they suggest that reality is not truly logical at all. However, the value of paradoxes is that they stimulate us to look more deeply for the logical connections in our intuitions and prove them in the language of logic or mathematics. While some paradoxes seem insurmountable, they also stimulate us to look beyond the use of the purely formal language of formal signs—used exclusively in logic and mathematics—to employ the representational language of empirical science and even the symbolic language of metaphysics (see Figure 1.1).

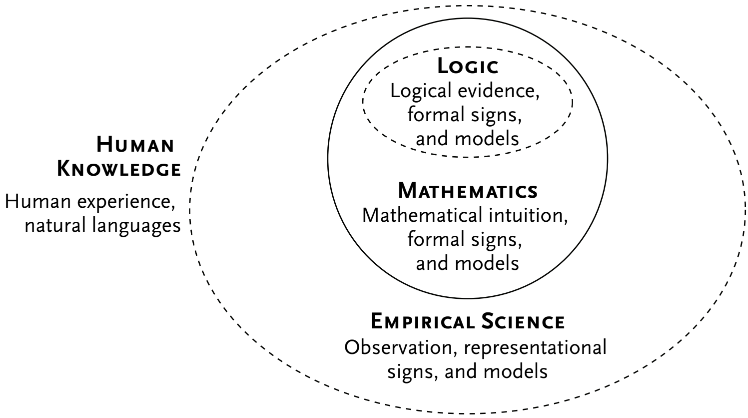


Figure 1.1

Representational Signs in Natural Science

The natural sciences begin with perceptions of the objects in the world, which is what separates them from the purely mental starting point of logic and mathematics, though natural science employs logic and mathematics as well. The empirical observations of natural science can be very sophisticated. Still, they are limited by the perceptions of the five senses. Once making their perceptions, scientists may certainly express them in natural languages, just as Copernicus spoke in Polish or German, but used Latin as academic language when he talked about his belief that the sun was stationary and the earth moved. Natural science ultimately seeks a high precision in its use of applied mathematics. Here mathematics becomes a privileged language; scientists understand each other and can conduct identical experiments despite their different national languages.

In practice, of course, the argument that natural science uses representational signs differently from the way logic or mathematics uses formal signs is a bit more subtle and complicated. The same signs can be used in either case. For instance, the signs E, m, and c in the equation $E = mc^2$ can be either formal or representational.

As formal signs they stand for elements of a mathematical structure, such as the system of real numbers (one of our more complex and inclusive sets of number systems). As representational signs, E, m, and c represent energy, mass, and speed of light. The difference between the two uses lies in the fact that, in the former, the semantics refer to mental objects (such as pure numbers) while, in the latter, the semantics refer to physical observations.

But we should emphasize again that logic and mathematics are at the core of natural science. Mathematics is indispensable in scientific research. The instruments that natural science uses to measure physical observations are designed based on mathematical theories. Moreover, logic and mathematics are not merely languages alone. They are the basic logical and mathematical intuitions that we cannot separate from our empirical sense experiences, and the bridge between those intuitions and our sense experiences has, in the history of science, been the building of scientific models.

Formal and Representational Models

When we create models, we have structures that help us imagine how the world works. Models are mediators between perception and theories. In science, these models designate and describe the relations between the parts of a given domain of discourse and the procedures we can use to analyze the topic of research. The domain of discourse contains all the elements to consider in a given model.

Naturally, science builds formal models of logic and mathematics, and it also builds representational models that describe empirical observations (such as the wooden solar-system model of early astronomers). The first (formal) is real, but it is purely conceptual and does not have to necessarily match the reality "out there," for it only needs internal consistency. A representational model, however, must somehow match the empirical reality that an ordinary person can observe.

The models may be used together, depending on the problem that science is trying to solve. I mentioned earlier the model of non-Euclidean geometry, which essentially speaks of something we call curved space, as distinct from the normal flat space of geometry. So a model of non-Euclidean geometry can be created to talk about a reality that is not known to us; it is speculative, in this sense. But also, a model of non-Euclidean geometry can be a representational picture of the physical reality spoken of by Einstein's theory of relativity, which is a mathematical theory verified by observing the curvature of light in space.

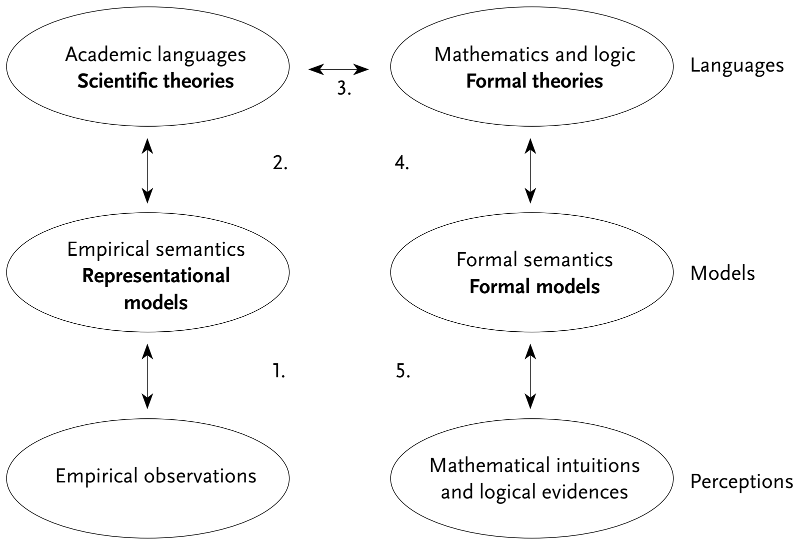
For another illustration of how these models interact, we can turn to the story of Nicholaus Copernicus in the sixteenth century. In his day, the earth-centered model of Ptolemaic astronomy—essentially based on Aristotle's physical model—had dominated Europe for more than a thousand years because this representational model succeeded in explaining what astronomers saw in the skies year after year. However, Copernicus offered a mathematical model that explained the physical observations just as well—and more simply than Ptolemy's model of circular orbits and epicycles.

Let's reconsider Einstein's work. His theory of relativity was a purely mathematical model since he was not an astronomer (and, indeed, neither was Copernicus for the most part). Einstein built in his mind a model that tried to create a logical system to explain the universe on scales that were too large to measure physically. Einstein's mathematical model was tested in the empirical world in 1919 when the English astronomer Arthur Eddington journeyed to the Principe Island during a total eclipse to measure, by photography, whether curved space bent light as Einstein's model argued. In turn, the result was that Einstein's mathematics could explain the photographs (rough and questionable as they were). Today, our common scientific language refers to curved space and the four dimensions of space-time—representational models based on the formal mathematical concepts.

Another example of mathematical models rivaling representational models came about in the debate over the smallest scales of matter, as was seen in the difference of opinion between two of our greatest modern physicists, Niels Bohr and his student Werner Heisenberg. They both tried to address the problem of the uncertainty of the position and velocity of particles at the level of quantum physics. Bohr preferred a visual model of the solar-system atom, and to resolve quantum uncertainty, he ended up with a somewhat paradoxical image of the atom: he said that we can have two opposing yet complementary, visual models, one with the particle as a wave and one with the particle as a tiny corpuscle. On the other hand, Heisenberg preferred a single mathematical model of probability to explain how a real particle can exist without clear coordinates in the physical atom—a model of how it can be a wave and a particle at the same time.

The lesson here is that in the history of science, empirical observations have usually been interpreted in more than one way. We refine our knowledge by trying to reconcile a mathematical model with a physical model that we can observe. We are most satisfied when this harmonizing of math and observation works very well, but in the mysteries and complexities of the universe, there is no guarantee. In Figure 1.2, we see how we begin with both physical and mathematical perceptions, and then work our way up to models and languages, and then try to reconcile the languages.

Though we speak in Figure 1.2 about mathematical, empirical, and academic languages, we are also bound by our natural languages. We are all born into different cultures that have long tried to describe what we perceive in the world, whether that description is spoken or written in German, Hindi, or Chinese. As noted earlier, scientific language helps us transcend our local languages, although the transcendence can seemingly never be complete. Even in scientific culture, a pluralism of points of view exists—a variety of cultural languages. These many scientific communities have their own journals, congresses, and rituals. At times each scientific community seems to be, in effect, a different country.



1. The empirical observations are explained through representational models of the reality. For example, Bohr's atom is a representational model of the interactions observed between the nucleus of the atom and the electrons. The representation of space-time in a four-dimensional space in the general theory of relativity is another example of a representational model of the empirical observations regarding the basic nature of matter.

2. The first explanations of the models of reality are expressed in an academic language. Academic languages are natural languages improved with some mathematical formalism. The use of mathematical formalisms in the explanation of the scientific models of reality cannot be separated from their explanation in a natural language.

3. The scientific theories can be explained by formal mathematical theories. For example, the laws of classical mechanics can explain the circular movement of the electrons around a nucleus in Bohr's atom, with the exception that only those orbits whose angular momentum is quantified are permitted. Another example of a formal theory that explains the models of the reality are the geometries of Hermann Minkowski through which the representational model of the theory of relativity is explained.

4. The formal mathematical theories are interpreted in formal models. For example, the formal theories of Minkowski are interpreted in a formal mathematical model of a non-Euclidean geometry.

5. The formal models of mathematics represent mathematical intuitions and logical evidences.

Figure 1.2

This situation reminds us of the difficulty in achieving pure objectivity in our perceptions, language, and model building. Even the purest formalism of logic cannot escape a degree of subjective interpretation. As we shall see next, even mathematicians disagree on what is absolutely logical.

Formalism and Objectivity

Although we may affirm that logic and mathematics are the most objective knowledge, they are not totally objective or totally independent of the knowing subject. The view of what is logical and what is mathematical depends on the principles of logic that we accept. Some communities of mathematicians accept certain logical principles that other communities do not accept.

Even though the logical processes of deduction—with formal syntax and formal semantics—are objective and automatic routines that a machine can execute, several different possible views are available of what logic is. Accepting one or another view of logic can depend on personal tastes and preferences for what counts as valid logic. One can assume different views of logic, but not at the same time if they are not consistent. Once a view is assumed, one must maintain consistency.

Surprising as it may seem, not all logicians accept the famous principle called the excluded middle, for example. This principle states that "all propositions are true or false." Classical logic is strongly rooted in the principle of the excluded middle. But other schools of thought in mathematics, such as the constructivist or intuitionist schools, do not accept this as an absolute premise. This debate over the excluded middle is a disagreement about the existence of what we call mathematical objects. The classical view is that to prove the existence of a mathematical object it is enough to derive a contradiction from the assumption of its nonexistence. According to the contrary (constructivist) view, one must find (or construct) any mathematical object in order to prove its existence.

The classical approach to mathematics strongly defends the principle of the excluded middle by reducing the contrary view to a position of absurdity, a common method of proof we call "reduction to the absurd" (or reductio ad absurdum). That is the idea behind the following scheme:

A or not A (Logical principle of excluded middle)
A implies B (Premise)
not B (Premise)

∴ not A (Conclusion)

In the case that A is true, B should be true, because A implies B. But B's existence with the premise not B would be absurd. Therefore if the logical principle of excluded middle is accepted, not A should be our conclusion.

In another example, using reduction to the absurd it can be proved that $\sqrt{2}$ is a number with infinite decimals. In order to prove this, it is sufficient to see that if $\sqrt{2}$ is a number with a finite number of decimals (A), then $\sqrt{2}$ could be written as the quotient of two whole numbers (B). Once we admit this fact (A implies B), it is enough to prove that $\sqrt{2}$ cannot be written as the quotient of two whole numbers (not B). Then, by the principle of excluded middle, we prove that $\sqrt{2}$ is a number with infinite decimals (not A).

This proof is valid in classical mathematics. Classical mathematicians admit the existence of sets with an infinite number of elements, whereas constructivist mathematicians do not admit the existence of such sets. What is evident for some is not so for others. Classical mathematicians believe that the existence of a mathematical object is proved if a contradiction is derived from its nonexistence, by reduction to the absurd. Constructivist mathematicians simply disagree. The members of the two groups chose their position based simply on personal preference, a valid enough reason for why people join one group or another, but a reason that is independent of the logical evidence.

The different views of mathematics—the classical and the constructivist—are based on different views of logic. When the Dutch mathematician L. E. J. Brouwer (d. 1966) stated that the principle of the excluded middle could not be applied in all cases, he was arguing for a new kind of logic. His statement was not supported by a logical deduction. Still, Brouwer persuasively justified his statement based on the meaning that the totality of mathematical activity had for him.

Another way to look at this fact of two or more systems of logic is to recognize that they all can be formalized into a rule-based system that can operate, for example, on a computer program. We can program a computer so that it executes classical deductions or constructivist deductions, or another type of deduction entirely, depending on the circumstances. However, when we speak of mathematicians as believers in one or another type of logic, then we are talking about personal preferences—and these typically appear as schools of thought.

This pattern of individual options and different groups of scientists illustrates that even when we speak of a universal activity such as logic or mathematics, these disciplines are not purely formal and mechanical. Logic does not easily transcend our normal human subjectivities. The student of logic can choose one logical principle or another, but there is no formal, logical argumentation—written in the heavens or on stone tablets, so to speak—that can help him to decide which one is best. The decision, in effect, is not a purely logical one.

That there are several views of logic and that these views depend on communities and their preferences can appear surprising. Indeed, the plurality of logics seems to contradict our impression that mathematical propositions and deductions have a high standard of certainty. Can logic and mathematics truly be universal languages?

Public Language

Despite this pluralism, logic and mathematics continue to be our most objective—and therefore privileged—instruments for public communication of knowledge. For a start, mathematical signs have the same meaning for everyone, whatever the circumstance. Within each mathematical system, all the elements are equal as causes of their relationships. Mathematical language has no room for the description of a final—that is, ultimate and directing—cause. Mathematics does not offer a First Cause or a Final Destination in the way it explains nature. This removes from mathematics the temptation to include the biases or agendas of human beliefs, making mathematics a useful public language.

Natural science also has its public language, thanks in great part to its relationship to logic and mathematics. Mathematics allows us to have a public language about physical, chemical, and biological realities. We can talk publicly about Isaac Newton's Second Law because its principle that force is proportional to mass and acceleration can be stated formally as F = ma. This principle can be explained in Chinese, Russian, or Swahili. The Darwinian theory of evolution by variation and natural selection has also entered our public language because scientists have observed and quantified many of these processes, and some of them have been written in mathematical language.

Every scientific observation cannot end up with a purely logical and mathematical presentation, of course. The world remains vast and paradoxical, not surrendering everywhere to human reason. So, in the case of the quantum physics of the atom, Niels Bohr has argued that we must simply use two kinds of mathematical objects to explain the apparent duality of particles as waves and corpuscles. Bohr calls this the principle of complementarity, presenting the case that we can rationally use two models to explain a single physical object in the universe. A particle can have two possible behaviors, and each of them can be explained by a mathematically consistent description. In physics, Bohr's principle of complementarity is famous for what it says about the limits of human perception. In modern culture, complementarity has also become the slogan of contemporary movements and beliefs that, often being antiscientific and postmodern, emphasize human creativity, paradox, and mysticism.

The complementarity spoken of in physics and in popular culture is not exactly the same as complementarity between the languages of logic, science, and metaphysics. Still, there is a similarity. The idea that the signs of logic and math positively complement the symbols of metaphysics has a parallel in physics, for in both cases, the two explanations are offered for one reality. Furthermore, reality transcends each of the explanations in both cases.

We mentioned earlier that the structure of mathematical language does not allow for the insertion of ultimate causes or ultimate outcomes. To find those ultimate causes we need the language of philosophy, metaphysics, or religion—we need a symbolic language. This language is nonmathematical and yet is able to create a self-consistent system that can speak to individuals and communities. In symbolic languages, we accept a plurality of ultimate meanings, causes, and reasons. We accept their plurality even though we live in the same mathematical and physical world.

Typically, we turn to such metaphysical language because neither formal logic and mathematics, nor empirical science and its models, can answer every question or solve every paradox.