Logic

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ROUTLEDGE

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Preface

I felt compelled to write an introductory textbook about formal logic for a number of reasons, most of which are pedagogic. I began teaching formal logic to undergraduates at the University of Edinburgh in 1985 and have continued to teach formal logic to undergraduates ever since. Speaking frankly, I have always found teaching the subject to be a particularly rewarding pastime. That may sound odd. Formal logic is widely perceived to be a difficult subject and students can and often do experience problems with it. But the pleasure I have found in teaching the subject does not derive from the anxious moments which every student experiences to some extent when approaching a first course in formal logic. Rather, it derives from later moments when self-confidence and self-esteem take a significant hike as students (many of whom will always have found mathematics daunting) realise that they can manipulate symbols, construct logical proofs and reason effectively in formal terms. The educational value and indeed the personal pleasure which such an achievement brings to a person cannot be overestimated. Enabling students to take those steps forward in intellectual and personal development is the source of the pleasure I derive from teaching formal logic. In these terms, however, the problem with existing textbooks is that they generally make too little contribution to that end.

For example, each and every year during my time at Edinburgh the formal logic class contained a significant percentage of arts students with symbol-based anxieties. More worryingly, these often included intending honours students who had either delayed taking the compulsory logic course, failed the course in earlier years or converted to Philosophy late. Many of these students were very capable people who only needed to be taught at a gentler pace or to be given some individual attention. Moreover, even the best of those students who were not so daunted by symbols regularly got into difficulties simply through having missed classes—often for the best of reasons. Given the progressive nature of the formal logic course these students frequently just failed to catch up. As a teacher, it was immensely frustrating not to be able to refer students (particularly those in the final category) to the textbook in any really useful way. The text we used was E.J.Lemmon's *Beginning Logic* [1965]. Undoubtedly, Lemmon's is, in many ways, an excellent text but the majority of students simply did not find it sufficiently accessible to be able to teach themselves from it. In all honesty, I think that this is quite generally the case with the vast majority of introductory texts in formal logic, i.e. inaccessibility is really only a matter of degree (albeit more so in the case of some than others). And this is no mere inconvenience for students and teachers. The underlying worry is that the consequent level of fail rates in formal logic courses might ultimately contribute to a decline in the teaching of formal logic in the universities or to a significant dilution of the content of such courses. For all of these reasons, I think it essential that we have a genuinely accessible introductory text which both covers the ground and caters to the whole spectrum of intending logic students, i.e. a text which enables students to teach themselves. That is what I have tried to produce here.

Logic covers the traditional syllabus in formal logic but in a way which may significantly reduce the kind of fail rates which, without such a text, are perhaps inevitable in compulsory courses in elementary logic offered within the Faculty of Arts. In the present climate, many faculties and, indeed, many philosophy departments consider such fail rates to be wholly unacceptable. Hence, the motivation to dilute the content of courses is obvious, e.g. by wholly omitting proof-theory. Personally, I believe that this cannot be a step in the right direction. In the last analysis, such a strategy either diminishes formal logic entirely or results in an unwelcome unevenness in the distribution of formal analytical skills among graduates from different institutions. I believe that the solution is to make available to students a genuinely accessible textbook on elementary logic which even the most anxious students in the class can use to teach themselves. Thus, *Logic* is not designed to promote my own view of formal logic as such or to promote the subject in any narrow sense. Rather, it is designed to promote formal logic in the widest sense, i.e. to make a subject which is generally perceived as difficult and inaccessible open and readily accessible to the widest possible audience.

To that end, the text is deliberately written in what I hope is a clear and user-friendly style. For example, formal statements of the rules of inference are postponed until the relevant natural deduction motivation has been outlined and an informal rule-statement has been specified. The text also makes extensive use of summary boxes of key points both during and at the end of chapters. Initial uses of key terms (and some timely reminders) are given in bold and such items are further explained in the glossary. Mock examination papers are also set at regular intervals in the text by way of dress rehearsal for the real thing. Given that accessibility is a crucial consideration, the pace of *Logic* is deliberately slow and indulgent. But this need not handicap either students or teachers. The text is exercise-intensive and brighter students can simply move to more difficult exercises more quickly. Moreover, the very point of there being such a text is to enable students to teach themselves. So teachers need not move as slowly as the text, i.e. the pace of the course may very well be deliberately faster than that of the text. The point is that the text provides the necessary back-up for slower students anyway. Further, those who miss classes can plug gaps for themselves, and while I have no doubt that certain students will still have problems with formal logic the text is specifically designed to minimise the potential for anxiety attacks.

I should also add that the text is tried and tested at least in so far as a desktop version has been used successfully at the University of Aberdeen for the past three academic sessions, over which, as I write, class numbers have trebled. The success of the text is reflected as much in course evaluation responses as in the pass rate for Formal Logic 1 (only one student failed Formal Logic 1 over sessions 1994–5 and 1995–6). Further, the pass rate for the follow-on course, Formal Logic 2, was 100 per cent in the first academic session and 95 per cent in the second academic session. Despite the increase in class numbers, pass rates in both courses remain very high and the contents of course evaluation forms suitably reassuring.

A certain amount of motivation for writing *Logic* also stems from some unease not just about the style but about the content of existing textbooks. For although many excellent texts are available, there is something of an imbalance in most. For example, while a number of familiar texts are quite excellent on semantic methods these tend to be wholly devoid of (linear or Lemmon-style) proof-theory. In contrast, texts such as Lemmon, for example, show a clear bias towards proof-theory and are not as extensive in their treatment of semantic concepts and methods as they might be. Indeed, certain texts in this latter category are either devoid of semantic methods at the level of quantificational logic or devote a very limited amount of space to such topics. Yet another group of familiar texts involves rather less in the way of formal methods generally. Ultimately, I think, such texts include too little in that respect for purposes of teaching formal logic to undergraduates. Hence, there is a strong argument for an accessible textbook which strikes a fair balance between syntactic and semantic methods. To that end, *Logic* combines a comprehensive treatment of proof-theory not just with the truth-table method but also with the truth-tree method. After all, the latter method is quite mechanical throughout both propositional logic and the monadic fragment of quantificational logic. Moreover, if that method is given sufficient emphasis at an early stage students can also be enabled to apply the method beyond monadic quantificational logic. Of course, in virtue of undecidability with respect to invalidity at that level, there is no guarantee of the success of any purely mechanical application of the truth-tree method, i.e. infinite branches and infinite trees are possible. But the application of the method at that level, together with examples of infinite trees and branches, vividly illustrates the consequences of undecidability to students and goes some way towards making clear just what is meant by undecidability. Finally, given that the method is also useful at the metatheoretical level, supplementing truth-tables with truth-trees from the outset seems a sound investment. In terms of content, then, the text covers the same amount of logical ground as any other text pitched at this level and, indeed, more than many.

In summary, *Logic* is primarily intended as a successful teaching book which students can use to teach themselves and which will enable even the most anxious students to grasp something of the nature of elementary logic. It is not intended to be a text which lecturers themselves will want to spend hours studying closely. Rather, it is intended to make a subject which is generally perceived as difficult and inaccessible open and easily accessible to the widest possible audience. In short, I hope that *Logic* constitutes a solution to what I believe to be a substantive teaching problem. However, if the text does no more than make formal logic accessible, comprehensible and above all useful to anxious students for whom it would otherwise have remained a mystery, then it will have fulfilled its purpose.

Paul Tomassi

1
How to Think Logically

I
Validity and Soundness

To study logic is to study **argument**. Argument is the stuff of logic. Above all, a logician is someone who worries about arguments. The arguments which logicians worry about come in all shapes and sizes, from every corner of the intellectual globe, and are not confined to any one particular topic. Arguments may be drawn from mathematics, science, religion, politics, philosophy or anything else for that matter. They may be about cats and dogs, right and wrong, the price of cheese, or the meaning of life, the universe and everything. All are equally of interest to the logician. Argument itself is the subject-matter of logic.

The central problem which worries the logician is just this: how, in general, can we tell good arguments from bad arguments? Modern logicians have a solution to this problem which is incredibly successful and enormously impressive. The modern logician's solution is the subject-matter of this book.

In daily life, of course, we do all argue. We are all familiar with arguments presented by people on television, at the dinner table, on the bus and so on. These arguments might be about politics, for example, or about more important matters such as football or pop music. In these cases, the term 'argument' often refers to heated shouting matches, escalating interpersonal altercations, which can result in doors being slammed and people not speaking to each other for a few days. But the logician is not interested in these aspects of argument, only in what was actually said. It is not the shouting but the sentences which were shouted which interest the logician.

For logical purposes, an *argument* simply consists of a sentence or a small set of sentences which lead up to, and might or might not justify, some other sentence. The division between the two is usually marked by a word such as 'therefore', 'so', 'hence' or 'thus'. In logical terms, the sentence or sentences leading up to the 'therefore'-type word are called **premises**. The sentence which comes after the 'therefore' is the **conclusion**. For the logician, an *argument* is made up of premises, a 'therefore'-type word, and a conclusion —and that's all. In general, words like 'therefore', 'so', 'hence' and 'thus' usually signal that a conclusion is about to be stated, while words like 'because', 'since' and 'for' usually signal premises. Ordinarily, however, things are not always as obvious as this. Arguments in daily life are frequently rather messy, disordered affairs. Conclusions are sometimes stated before their premises, and identifying which sentences are premises and which sentence is the conclusion can take a little careful thought. However, the real problem for the logician is just how to tell whether or not the conclusion really does follow from the premises. In other words, when is the conclusion a **logical consequence** of the premises?

Again, in daily life we are all well aware that there are good, compelling, persuasive arguments which really do establish their conclusions and, in contrast, poor arguments which fail to establish their conclusions. For example, consider the following argument which purports to prove that a cheese sandwich is better than eternal happiness:

1. Nothing is better than eternal happiness.
2. But a cheese sandwich is better than nothing.

Therefore,

3. A cheese sandwich is better than eternal happiness.[1]

Is this a good argument? Plainly not. In this case, the sentences leading up to the 'therefore', numbered '1' and '2' respectively, are the premises. The sentence which comes after the 'therefore', Sentence 3, is the conclusion. Now, the premises of this argument might well be true, but the conclusion is certainly false. The falsity of the conclusion is no doubt reflected by the fact that while many would be prepared to devote a lifetime to the acquisition of eternal happiness few would be prepared to devote a lifetime to the acquisition of a cheese sandwich. What is wrong with the argument is that the term 'nothing' used in the premises seems to be being used as a name, as if it were the name of some other thing which, while better than eternal happiness, is not quite as good as a cheese sandwich. But, of course, 'nothing' isn't the name of anything.

In contrast, consider a rather different argument which I might construct in the process of selecting an album from my rather large record collection:

1. If it's a Blind Lemon Jefferson album then it's a blues album.
2. It's a Blind Lemon Jefferson album.

Therefore,

3. It's a Blues album.

Now, this argument is certainly a good argument. There is no misappropriation of terms here and the conclusion really does follow from the premises. In fact, both the premises and the conclusion are actually true; Blind Lemon Jefferson was indeed a bluesman who only ever made blues albums. Moreover, a little reflection quickly reveals that if the premises are true the conclusion must also be true. That is not to say that the conclusion is an eternal or **necessary truth,** i.e. a sentence which is always true, now and forever. But if the premises are actually true then the conclusion must also be actually true. In other words, this time, the conclusion really does follow from the premises. The conclusion is a logical consequence of the premises. Moreover, the necessity, the force of the 'must' here, belongs to the relation of consequence which holds between these sentences rather than to the conclusion which is consequent upon the premises. What we have discovered, then, is not the necessity of the consequent conclusion but the necessity of logical consequence itself.

In logical terms the Blind Lemon Jefferson argument is a **valid** argument, i.e. quite simply, if the premises are true, then the conclusion must be true, on pain of contradiction. And that is just what it means to say that an argument is valid: whenever the premises are true, the conclusion is guaranteed to be true. If an argument is valid then it is impossible that its premises be true and its conclusion false. Hence, logicians talk of validity as preserving truth, or speak of the transmission of truth from the premises to the conclusion. In a valid argument, true input guarantees true output.

Is the very first argument about eternal happiness and the cheese sandwich a valid argument? Plainly not. In that case, the premises were, indeed, true but the conclusion was obviously false. If an argument is valid then whenever the premises are true the conclusion is guaranteed to be true. Therefore, that argument is **invalid**. To show that an argument fails to preserve truth across the inference from premises to conclusion is precisely to show that the argument is invalid.

The Blind Lemon Jefferson example also illustrates the point that logic is not really concerned with particular matters of fact. Logic is not really about the way things actually are in the world. Rather, logic is about argument. So far as logic is concerned, Blind Lemon Jefferson might be a classical pianist, a punk rocker, a rapper, or a country and western artist, and the argument would still be valid. The point is simply that:

If it's true that:	If it's a Blind Lemon Jefferson album then it's a blues album.
And it's true that:	It's a Blind Lemon Jefferson album.
Then it must be true that:	It's a blues album.

However, if one or even all of the premises are false in actual fact it is still perfectly possible that the argument is valid. Remember: validity is simply the property that if *the* premises are all true *then* the conclusion must be true. Validity is certainly not synonymous with truth. So, not every valid argument is going to be a good argument. If an argument is valid but has one or more false premises then the conclusion of the argument may well be a false sentence. In contrast, valid arguments with premises, which are all actually true sentences must also have conclusions which are actually true sentences. In Logicspeak, such arguments are known as **sound** arguments. Because a sound argument is a valid argument with true premises, the conclusion of every sound argument must be a true sentence. So, we have now discovered a very important criterion for identifying good arguments, i.e. *sound arguments are good arguments.* But surely we can say something even stronger here. Can't we simply say that sound arguments are definitely, indeed, definitively good arguments? Well, this is a controversial claim. After all, there are many blatantly circular arguments which are certainly sound but which are not so certainly good.

For example, consider the following argument:

1. Bill Clinton is the current President of the United States of America.

Therefore,

2. Bill Clinton is the current President of the United States of America.

We can all agree that this argument is valid and, indeed, sound. But can we also agree that it is really a good argument? In truth, such arguments raise a number of questions some of which we will consider together later in this text and some of which lie beyond the scope of a humble introduction to what is ultimately a vast and variegated field of study. For present purposes, it is perfectly sufficient that you have a grasp of what is meant by saying that an argument is *valid* or *sound*.

To recap, sound arguments are valid arguments with true premises. A valid argument is an argument such that if the premises are true then the conclusion must be true. Hence, the conclusion of any sound argument must be true. But do note carefully that validity is not the same thing as truth. Validity is a property of arguments. Truth is a property of individual sentences. Moreover, not every valid argument is a sound argument. Remember: a valid argument is simply an argument such that if the premises are true then the conclusion must be true. It follows that arguments with one or more premises which are in fact false and conclusions which are also false might still be valid none the less. In such cases the logician still speaks of the conclusion as being validly drawn even if it is false. On false conclusions in general, one American logician, Roger C.Lyndon, prefaces his logic text with the following quotation from Shakespeare's *Twelfth Night:* 'A false conclusion; I hate it as an unfilled can.'[2] That sentiment is no doubt particularly apt as regards a false conclusion which is validly drawn. None the less, it is perfectly possible for a false conclusion to be validly drawn. For example:

1. If I do no work then I will pass my logic exam.
2. I will do no work.

Therefore,

3. I will pass my logic exam.

So, not all valid arguments are good arguments, but the important point is that even though the conclusion is false, the argument is still valid, i.e. if its premises really were true then its conclusion would also have to be true. Hence, the conclusion is validly drawn from the premises even though the conclusion is false.

Moreover, valid arguments with false premises can also have actually true conclusions. For example:

1. My uncle's cat is a reptile.
2. All reptiles are cute, furry creatures.

Therefore,

3. My uncle's cat is a cute, furry creature.

This time both premises are false but the conclusion is true. Again, the argument is valid none the less, i.e. it is still not possible for the conclusion to be false if the premises are true. Further, while we might not want to say that this particular argument is a good one, it is worth pointing out that there are ways in which we can draw conclusions from a certain kind of false sentence which leads to a whole class of arguments which are obviously good arguments. We will consider just this kind of reasoning in some detail later in Chapter 3. For now, remember that validity is not synonymous with truth and that validity itself offers no guarantee of truth. If the premises of a valid argument are true then, certainly, the conclusion of that argument must be true. But just as a valid argument may have true premises, it may just as easily have false premises or a mixture of both true and false premises. Indeed, valid arguments may have any mix of true or false premises with a true or false conclusion excepting only that combination of true premises and false conclusion. Only sound arguments need have actually true premises and actually true conclusions. Therefore, soundness of argument is the criterion which takes us closest to capturing our intuitive notion of a good argument which genuinely does establish its conclusion. Whether we can simply identify soundness of argument with that intuitive notion of good argument remains controversial. But what is surely uncontroversial is that validity and soundness of argument are integral parts of any attempt to make that intuition clear.

II
Deduction and Induction

In the ordinary business of daily life (and particularly in films about Sherlock Holmes) we generally find the term 'deduction' used in a very loose sense to describe the process of reasoning from a set of premises to a conclusion. In contrast, logicians tend to use the same term in a rather narrower sense. For the logician, **deductive argument** is valid argument, i.e. validity is the logical standard of deductive argument. Hence, you will frequently find valid arguments referred to as *deductively valid* arguments.

In Logicspeak the premises of a valid argument are said to **entail** or **imply** their conclusion and that conclusion is said to be *deducible* from those premises. But deduction is not the only kind of reasoning recognised by logicians and philosophers. Rather, deduction is one of a pair of contrasting kinds of reasoning. The contrast here is with **induction** and **inductive argument**. Traditionally, while deduction is just that kind of reasoning associated with logic, mathematics and Sherlock Holmes, induction is considered to be the hallmark of scientific reasoning, the hallmark of scientific method. For the logician deductive reasoning is valid reasoning. Therefore, if the premises of a deductive argument are true then the conclusion of that argument must be true, i.e. validity is truth-preserving. But validity is certainly not the same as truth and deduction is not really concerned with particular matters of fact or with the way things actually are in the world. In sharp contrast, and just as we might expect of scientists, induction is very much concerned with the way things actually are in the world.

We can see this point illustrated in one rather simple kind of inductive argument which involves reasoning, as we might put it, from the particular to the general. Such arguments proceed from a set of premises reporting a particular property of some specific individuals to a conclusion which ascribes that property to every individual, quite generally. Inductive arguments of this kind proceed, then, from premises which need be no more than records of personal experience, i.e. from *observation-statements*. These are **singular sentences** in the sense that they concern some particular individual, fact or event which has actually been observed. For example, suppose you were acquainted with ten enthusiastic and very industrious logic students. You might number these students 1, 2, 3 and so on and proceed to draw up a list of premises as follows:

1. Logic student #1 is very industrious.
2. Logic student #2 is very industrious.
3. Logic student #3 is very industrious.
4. Logic student #4 is very industrious.

.

.

10. Logic student #10 is very industrious.

In the light of your rather uniform experience of the industriousness of students of logic you might well now be inclined to argue thus:

Therefore,

11. Every logic student is very industrious.

Arguments of this kind are precisely inductive. From a finite list of singular observation-statements about particular individuals we go on to infer a general statement which refers to all such individuals and attributes to those individuals a certain property. For just that reason, the great American logician Charles Sanders Peirce described inductive arguments as 'ampliative arguments', i.e. the conclusion goes beyond, 'amplifies', the content of the premises. But, if that is so, isn't there a deep problem with induction? After all, isn't it perfectly possible that the conclusion is false here even if we know that the premises are true? Certainly, the industriousness of ten logic students does not guarantee the industriousness of every logic student. And, indeed, if that is so, induction is invalid, i.e. it simply does not provide the assurance of the truth of the conclusion, given the truth of the premises, which is definitive of deductive reasoning. But aren't invalid arguments always bad arguments? Certain philosophers have indeed argued that that is so.[3] On the other hand, however, couldn't we at least say that the premises of an inductive argument make their conclusion more or less likely, more or less probable? Perhaps a list of premises reporting the industriousness of a mere ten logic students does not make the conclusion that all such students are industrious highly probable. But what of a list of 100 such premises? Indeed, what of a list of 100,000 such premises? If the latter were in fact the case, might it not then be highly probable that all such students were very industrious?

Many philosophers have considerable sympathy with just such a probabilistic approach to understanding inductive inference. And despite the fact that induction can never attain the same high standard of validity that deduction reaches, some philosophers (myself included!) even go so far as to defend the claim that there are good inductive arguments none the less. We cannot pursue this fascinating debate any further here. For, if there are good inductive arguments, these have a logic all of their own. Interested parties can find my own account of the logic of scientific reasoning and a defence of the idea that there can be good inductive arguments in my paper 'Logic and Scientific Method'.[4] For present purposes, it is sufficient to appreciate that inductive reasoning is not valid reasoning.

III
The Hardness of the Logical 'Must'

In the previous section we again noted that invalid arguments fail to establish the truth of their conclusions even when the premises of such an argument are actually true. In contrast, given that the premises are true, the conclusion of any valid argument must be true. So, what is it about a valid argument with true premises which compels us to accept the conclusion of that argument? In the course of ordinary daily life, we find that many different things can compel us to accept the conclusion of an argument as a consequence of its premises: large persons of a violent disposition will often secure agreement to the conclusions of their arguments, for instance. But it is not the threat of violence that compels us to accept the logicians' conclusions. Rather, it is **logical force,** the force of reason. Again, we can appeal to the definition of validity to cash out quite what logical force comes to: valid arguments establish their conclusions conditionally upon the truth of all their premises. Consider a very clear example of valid argument:

1. All human beings are mortal.
2. Prince is a human being.

Therefore,

3. Prince is mortal.

Of course, the premises may not be true. Some human beings may be immortal. Prince may not be a human being. But if all human beings are mortal and if Prince is a human being then it must follow that Prince is mortal. So, once I have accepted the truth of the premises here I am forced to accept the truth of the conclusion. Why? Because if I do not accept the truth of the conclusion having accepted the truth of the premises then I have blatantly contradicted myself. In this case the contradiction consists in believing that all human beings are mortal and that Prince is a human being who is not mortal. It cannot be rational to believe contradictions. Therefore, I must accept the truth of the conclusion, on pain of irrationality. (Check that this is also the case in each of the valid examples given earlier.) The hardness of the logical 'must' is the hardness of reason. Logical force is the force of reason.

This point gives some insight into the traditional definition of logic as 'the science of thought', the study of the rationality of thinking. In the last analysis, we ourselves may not want to defend quite such a subjective, psychologistic definition of the subject, but supposing that we can identify the laws of logic and represent them mathematically (we shall see later that we can) we can at least make clear sense of George Boole's account of the laws of logic, in his *Mathematical Analysis of Logic* [1847]:

> The laws we have to examine are the laws of one of the most important of our mental faculties. The mathematics we have to construct are the mathematics of the human intellect.

IV
Formal Logic and Formal Validity

Earlier, I noted that logic is not really about matters of fact or particular cause and effect relations but is concerned instead with validity which is independent of any such worldly, factual or, in philosophical terms, **empirical** matters. Recall the Blind Lemon Jefferson example. Perhaps you find it unconvincing. You might think that Blind Lemon may have been a milkman rather than a bluesman. But if you substitute the name of your own favourite blues performer the argument at once appears convincing and sound.

In one sense, it really doesn't matter which particular performer's name I actually used: we can legitimately substitute the name of any performer or any band and still retain a valid argument. Indeed, it needn't even be a Blues band. What is important is not the name of the band but the pattern of argument. When you substitute the name of your favourite band for 'Blind Lemon Jefferson' something changes. But something also remains the same: the pattern or structure of the argument. In fact, the only thing that changes is the particular name used in each sentence. The type of sentence is the same and the overall structure of the argument is the same. What is in common between your favourite example and my favourite example is the **logical form** of the argument. What is important to the formal logician is not the content of the argument but its form.

Change the name of the supposed bluesman as you will, the form of argument remains exactly the same. So, logic is not really about particular matters of fact and it is not really about particular bluesmen either. Rather, formal logic is about **argument-forms** (logically enough). Most importantly, as we shall see, the formal logician can use the notion of logical form to investigate the concept of validity. For example, consider the Blind Lemon Jefferson argument. Clearly, it is a valid argument. If the premises are true the conclusion must be true. But, as we have just seen, we can change the name of the bluesman or even substitute the name of any band and still get a valid argument. Moreover, as you will see, we can in fact change the most basic sentences which make up the premises and conclusion and still produce a valid argument. What makes this possible is the fact that the validity of this argument does not depend on particular matters of fact or particular bluesmen. Rather, it is the form and structure of the sentences in the argument and the relations between those sentences which guarantee that we cannot have true premises with a false conclusion in any such argument.

It follows that any argument of that particular logical form will also be a valid argument. Thus, formal logicians use the notion of logical form to investigate the concept of validity. Indeed, many formal logicians will now encourage us to replace the intuitive definition of validity we have been working with so far in favour of the following purely **formal definition** of validity:

> An argument is valid if, and only if, it is an instance of a valid logical form.

Hence, formal logic is fundamentally concerned with valid logical forms of argument. Formal logic, we might say, investigates formal validity. Further, it can be argued that the intuitive or **modal definition** is not an entirely adequate one (the term ‘modal’ is appropriate here because it refers to the notion of necessity, the ‘must’ element of our definition). For example, consider the following argument carefully:

1. Snow is white.

Therefore,

2. 1+1=2.

This argument seems to satisfy the modal definition. But the conclusion surely is not a logical consequence of the premise. However, the argument is not an instance of any valid logical form of argument. Therefore, it is formally invalid. So, perhaps we should adopt a purely formal definition. But is validity always a formal matter? In all honesty, it is not entirely clear that it is. Some arguments are intuitively valid, i.e. valid in terms of the modal definition, even though they seem to exhibit no valid logical form. Here is an example:

1. The Statue of Liberty is green.

Therefore,

2. The Statue of Liberty is coloured.

In fact, although this particular argument is intuitively valid, it is, as we shall see, an instance of at least one obviously invalid logical form. Moreover, the problem here is a deep, intractable one for there does not seem to be any way in which we can faithfully amend the sentences composing the argument which would result in the argument becoming formally valid. For example, consider another case which might seem similar:

1. All unmarried men are unmarried men.

Therefore,

2. All bachelors are unmarried men.

Again, the problem is precisely that the argument is an instance of an invalid logical form. In this case, however, the premise is obviously a necessary or logical truth while the conclusion is not obviously so. But the terms 'bachelors' and 'unmarried men' are synonyms. And if we substitute the term 'bachelors' in the conclusion with 'unmarried men' we generate the following argument:

1. All unmarried men are unmarried men.

Therefore,

2. All unmarried men are unmarried men.

This argument is obviously circular but it is also obviously valid and sound, and, crucially, it can now be shown to be an instance of a valid logical form. So, while we cannot honestly say that the first version of the argument is an instance of a valid logical form we can say that it is an argument which will become an instance of a valid logical form after appropriate substitution of synonyms.

But now consider the example about the Statue of Liberty again. In this case, synonym substitution is not legitimate. The terms 'green/coloured' do not represent a synonym pair. Certainly all green things are coloured. But not all coloured things are green! What does this prove? In the last analysis, it may well prove that validity cannot be completely explained in purely formal terms. In truth, however, this again is a matter of some controversy. As we shall see, the notion of logical form is not an absolute one, i.e. the same argument can be an instance of more than one form. Perhaps we have simply failed to find that valid form of which our argument about the Statue of Liberty is an instance. Perhaps not. Alternatively, we could simply adopt the formal definition and find another term to describe those arguments which seem to slip through the formal net, as it were. Be that as it may, our hand is not forced here. And so, although we should note this important controversy carefully, we will not abandon the intuitive or modal conception of validity we have been working with to date in favour of a purely formal definition.

While it is not incumbent upon us to resolve the controversy about the definition of validity here, it is crucially important to appreciate that logic is a discipline which contains many fascinating and important controversies. Indeed, within formal logic itself there is even room for disagreement about the validity of the argument-forms sanctioned by a given formal system, i.e. about the correctness of the formal system itself. Notably, it is precisely that possibility which Captain James T.Kirk regularly overlooks in the well-known television programme *Star Trek* when he accepts Mr Spock's allegations of illogicality. What Kirk fails to realise is that there exist a number of distinct, competing systems of formal logic which sanction distinct sets of argument-forms. Hence, in one sense, there is no single correct logic. It follows that a proper formal definition of validity is only fully specified for a particular set of forms and so one can only really make an informed judgement once that set has been laid out. The particular system of formal logic upon which we will focus in this text is the traditional or **classical logic** which was formulated first. In all honesty, alternative systems are best (and most easily) understood as revisions of that traditional system which arise from both formal and philosophical thinking about classical logic. So, it is the classical system to which we should devote ourselves first.

Finally, in the light of the possible limitation to the adequacy of the formal definition of validity considered above, one might wonder whether logicians should concentrate on purely formal logic. Would we do better to pursue our logical investigations informally?

This is a ticklish question. Part of the answer to it is just that formal logic embodies many of the very standards we would need to pursue our informal investigations! So, the best student of informal logic will be the one who has first mastered formal logic. Moreover, even if we do accept that the formal logician cannot completely explain validity in purely formal terms, classical formal logic captures a huge class of valid forms none the less. Therefore, formal logic remains a crucially important and highly effective means of investigating the concept of validity.

V
Identifying Logical Form

In the previous section, we noted that the formal analysis of validity may be incomplete. However, we should not be too daunted by that fact. The job of the formal logician consists in unearthing valid forms of argument and the sheer extent to which the formal logician is able to do that job effectively is astonishing. But just how far does formal validity go? To provide an answer to that very question is the purpose of this book. And where better to begin than with old Blind Lemon Jefferson? In the Blind Lemon Jefferson case, the structure of the sentences and the pattern of argument are very easy to see. The first sentence is clearly an 'If…then—' sentence:

1. *If* it's a Blind Lemon Jefferson album *then* it's a blues album.
2. It's a Blind Lemon Jefferson album.

Therefore,

3. It's a blues album.

Stripped bare, as it were, this argument has the form:

1. *If…then—-*
2. …

Therefore,

3. —

Looked at in this way, there are two gaps or *places* to be filled in the first premise, i.e. '…' (pronounced 'dot, dot, dot') and '—' (pronounced 'dash, dash, dash'). So, the structure of the first premise is just: If…then —. In the Blind Lemon Jefferson case, the first gap is filled in by the sentence 'It's a Blind Lemon Jefferson album' which is precisely the same sentence as the second premise. The second gap is filled by the sentence 'It's a blues album' which is precisely the same sentence as the conclusion: 'It's a blues album.' But now it is clear that as long as we stick precisely to the same form of argument we could have used any two sentences and we would still have had a valid argument. Hence, any argument of this form is bound to be valid, i.e. any argument consisting of any sentences in those relations must be valid. And that is just what it means to say that a form of argument is valid. So, not only is logic not concerned with particular matters of fact, or particular bluesmen, it is not even concerned with particular sentences.

Logically enough, formal logic is fundamentally concerned with forms of argument. Forms of argument are really **argument-frames** or *schemas*, i.e. patterns of inference with gaps which, for present purposes, can be filled using any particular sentences we choose to pick, provided only that we do complete the form exactly. Since it doesn't matter which particular sentences are involved in a given form it would be useful to have symbols which just marked the gaps, place-markers, for which we could substitute any sentence. This would save us writing out whole sentences, or marking gaps with '…' and '—'.

In algebra, mathematicians generally use the symbols '*x*' and '*y*' to stand for any numbers. Because such symbols mark a place for *any* number they are called **variables**. But the logician has no need to borrow these variables. Logicians have their own variables. In the present context, the logicians' variables mark places not for numbers but for sentences or, in more traditional logical terms, **propositions**. A proposition is thought of as identical with the meaning or sense of a sentence rather than with the actual sentence itself. So, intuitively, two different sentences which are really just two different ways of saying exactly the same thing are said to express one and the same proposition. For example, the following two sentences would be said to express only one proposition:

1. Edinburgh lies to the north of London.
2. London lies to the south of Edinburgh.

Talking of propositions rather than sentences can constitute a linguistic economy and many find the concept of a proposition both natural and intuitive. The idea is not uncontroversial and a fascinating debate has grown up around the simple questions of whether there are such things as propositions and, if so, just what kind of thing they might be. Those interested in these questions would do well to read the first chapter of W.V.O. Quine's *Philosophy of Logic*,[5] though, unfortunately, such questions lie beyond the scope of the present text.

For present purposes, we will bypass this particular debate by simply taking the lower-case letters '*p*', '*q*', '*r*' and so on as being **sentential variables,** i.e. variables whose values are simply well-formed sentences.

As schematic letters, sentential variables make it very easy to express precisely the bare pattern or logical form of an argument. For example, we can easily represent the logical form of the Blind Lemon Jefferson argument, as follows:

1. If p and q
2. p

Therefore,

3. q

When formalising a given argument, the crucial point to note is just that the same variable must mark a place for the same sentence throughout that formalisation. (The example about the logic examination also has exactly the same logical form we have just identified. This time, however, the variables 'p' and 'q' mark places for two different sentences—work out which.)

Above all, the formal logician is interested in forms of argument. Therefore, the central problem for the logician becomes: how are we to tell good forms of argument from bad forms of argument? In other words, how do we distinguish valid forms from invalid forms? According to the formal logician, a form of argument is valid if, and only if, every particular instance of that argument-form is itself valid. Thus valid argument forms are patterns of argument which, when followed faithfully, should always lead us to construct particular valid arguments as instances. For obvious reasons, this is known as the **substitutional criterion of validity**. I will offer a precise definition later but for the moment here is an analogy. Consider the following simple algebraic equation: $2x+2x=4x$. For every particular value of the variable x in this equation, be it apples, pears or double-decker buses, it will always be true that two of them added to another two will add up to four in total. Analogously, for any valid argument form, every particular argument which really is a **substitution-instance** of that form will itself be a valid argument, whether it concerns Blind Lemon Jefferson, passing your exams or anything else.

Unfortunately, we may have to recognise another limitation to the purely formal account later. Certain logicians have argued that the substitutional criterion is ultimately incomplete, just as it stands. These logicians allege that the criterion turns out to sanction as valid certain forms which have obviously invalid instances.[6] If that is so, we must indeed recognise another limitation to the purely formal account. This particular allegation raises a number of questions which, again, lie beyond the scope of the present text. Be that as it may, it should now be clear that formal logic is fundamentally concerned with valid forms of argument. Indeed, the traditional or classical logic which we will consider together in this text is one attempt to identify and elucidate all the valid forms of argument.

As such, logic is the study of the structure and principles of reasoning and of the nature of sound argument. But it is important to note that logicians need not always arrive at those principles of deductive inference which form the subject-matter of their field of study by collecting data about the way people actually argue. Boole's rather traditional definition might well give that impression but the relation between formal logic and actual argument is more complex. The two interact. As we have already seen, logic has traditionally been described as the science of thought. If it is a science, however, logic is a theoretical science, not an empirical science.

A good way of elucidating this distinction is with an analogy to games. Chess, in particular, is an excellent example. Logic, in the analogy, is like the rules of the game of chess, the rules of play which govern the game and define what chess is. The relation between the logical principles of deductive inference and the actual arguments people use, the inferences made by 'the person in the street' or 'the person on the Clapham omnibus', as it used to be said, is analogous to the relation between the rules of the game of chess and the actual playing of the game. The famous Austrian philosopher Ludwig Wittgenstein, in Remark 81 of his *Philosophical Investigations*, quotes a definition of logic by the mathematical logician F.P.Ramsey as a 'normative science'.[7] This is a good description which allows us to develop (and update) our definition of formal logic: formal logic constitutes a set of rules and standards, ideals of inference, or *norms*, independent of the thinking of any actual individual, in terms of which we appraise and assess the actual inferences which individuals make. So, in its concern with the ways in which people do actually argue, logic is *scientific* but in so far as logic provides standards of argument it is also *normative*.

To sum up, formal logic is fundamentally concerned with the form and structure of arguments and not, primarily, with their content. In terms of the chess analogy, it is the study of the rules of the game, not of the strategies of any particular player.

VI
Invalidity

According to the modal definition of validity an argument is valid if, and only if, whenever its premises are true its conclusion must also be true, i.e. if, and only if, it would be impossible for its premises to be true and its conclusion false. It follows logically that no valid argument can have true premises and a false conclusion. Indeed, to show that an argument is invalid is precisely to show a way in which that argument could have true premises and a false conclusion. In general then, an argument is invalid if it is such that its premises could all be true and its conclusion false.

Therefore, in order to demonstrate that a given argument is invalid it is sufficient to indicate that even if the premises are true the conclusion is actually false, or could be false, while the premises were true. For example, the former is precisely what is the case as regards the very first argument concerning the cheese sandwich which we considered at the outset of this chapter. Therefore, that argument is invalid. However, particular arguments are of interest to the formal logician only in so far as they exhibit logical forms of argument. Above all, logic is the study of forms of argument. Therefore, the fundamental question at this stage is just: how do we show that a given form of argument is invalid?

Recall the substitutional criterion: a form of argument really is valid if, and only if, every substitution-instance of that form is itself a valid argument. It follows that an argument-form is valid if, and only if, it is not the case that there is any instance of that form which has true premises and a false conclusion.

In order to demonstrate that a given form of argument is invalid, then, it is sufficient to exhibit some particular example of the form in question that could have actually true premises and a false conclusion. Any such invalid particular instance of a form is known as a **counterexample** to that form. The method of proving invalidity by means of a counterexample is known as **refutation by counterexample**. In practice, it is a devastatingly effective argumentative technique. Consider the following argument-form:

1. If *p* then *q*
2. *q*

Therefore,

3. *p*

Here is a counterexample to the form concerning my black cat, Zebedee (for the purposes of many of the examples in this book it is worth bearing in mind that I am the proud owner of two small black cats called Tiffin and Zebedee):

1 If all cats are black then Zebedee is black.
2. Zebedee is black.

Therefore,

3. All cats are black.

Now check for yourself:

1. That the argument is an instance of the logical form in question.
2. That the premises are actually true in this case.
3. That the conclusion is actually false.

Consider another argument-form:

1. If *p* then not *q*
2. Not *p*

Therefore,

3. *q*

Here is a counterexample to this form:

1. If Tiffin is a dog then it is not the case that Tiffin is an elephant.
2. Tiffin is not a dog.

Therefore,

3. Tiffin is an elephant.

Again, check for yourself:

1. That the argument really is an instance of the form in question.
2. That the premises are true.
3. That the conclusion is false.

It is important to note that I am not using any algorithm, i.e. any step-by-step, mechanical decision-procedure, to produce these counterexamples. At this stage, producing actual counterexamples requires art and imagination (and a fair bit of practice!). So, don't worry if you cannot come up with your own examples. It is sufficient that you understand the particular examples given.

VII
The Value of Formal Logic

Many students are very daunted by the prospect of a first logic course and feel extremely anxious at the outset of their course. If that is your experience, you can at least rest assured that you are not alone in your angst. In fact, many formal logicians will themselves have felt just as you do at this stage in the inquiry. So, the point is not simply that you have company but rather that you are in good company. Moreover, I feel sure that you will have found at least some of the ground we have covered together in the present chapter both accessible and intuitive. It is important to realise why that should be so. The point is a very simple one: as a matter of fact, we do all reason logically in daily life perfectly successfully and in ways which are often just as complex as those we will consider together in the present text.

As we noted earlier, formal logic is the study of the rules of the game rather than the strategy of the individual player. None the less, we should never lose sight of the fact that we do all reason logically in ordinary life. As I might put it, we all do on at least a part-time basis what the formal logician does full-time. And that fact is underwritten by a still more fundamental point: human beings are born with a natural ability to argue, to reason and to think logically. In his later work, Wittgenstein rightly made much of the simple point that many of our attitudes and abilities, ways of acting and ways of reacting, follow from the *form of life* we share just as human beings. Fortunately, the ability to argue and to reason logically is part of that natural legacy.

To realise that the study of formal logic is not really a matter of memorising and applying daunting mechanical rules but is rather a reflective study of how well we can all naturally reason at our very best is to realise the true value of the study of formal logic. The logician A.A.Luce puts this point very well when he notes that:

> the study acquires a new status and dignity when viewed as a conscious awakening of an unconscious natural endowment.[8]

As this book develops, our concern with argument will inevitably focus upon forms of argument rather than the particular arguments which we might construct day to day in a natural language such as English. But we should never lose sight of the fact that formal logic has its roots in just such natural language arguments and has enormous applicability to arguments in natural language, quite generally.

For the philosopher in particular, formal logic is a potentially devastating weapon which can and should be deployed in debate. If you lose sight of the applicability of formal logic to natural language arguments then you will miss out on a crucial aspect of the power and value of formal logic and much of its excitement. Something of the applicability of formal logic should be clear already. After all, the classical logician has provided us with some powerful tools for telling good arguments from bad, for identifying logical forms of argument, and for exposing the invalidity both of particular arguments and of argument-forms.

It is often difficult to exploit formal logic in debate but when it can be brought to bear it can be extremely effective. There is a famous story of a debate between the eminent classical logician Bertrand Russell and Father Frederick Copleston which clearly illustrates just how useful knowledge of formal logic can be. The debate in question concerned a particular argument known as the ‘cosmological argument’. This argument is one of the traditional arguments (we cannot say 'proof', for that begs the question) for the existence of God. The argument moves from the premise that every event has a cause to the conclusion that there must, at some point, be a first cause and this is God. Father Copleston defended the cosmological argument in the debate. What is of interest to us here is the way in which Russell attacked the argument.

In effect, Russell represented the cosmological argument as follows:

1. Every event has a cause.

Therefore,

2. Some event is the cause of every event.

Next, Russell tried to identify the form of the argument, thought hard about the validity of that form, and then produced the following counterexample:

1. Everyone has a mother.

Therefore,

2. Someone is the mother of everyone.

What Russell attempted to show is that the cosmological argument is invalid because it is an instance of an invalid form of argument. The form of reasoning which Russell highlights is certainly an invalid one. Indeed, arguments of that form exemplify a well-known fallacy, the **quantifier switch** or **quantifier shift fallacy**. Stating the form of this particular fallacy requires more logical machinery than is available to us at this stage. But, as we will see in Chapter 5, the form of the fallacy certainly can be made explicit. However, even if Russell has shown that the argument is an instance of that invalid form this does not prove that the cosmological argument is invalid. As we noted in Section IV, a particular argument may be an instance of more than one form. So, on one level of analysis, the argument might well be shown to be an instance of an invalid form but if we are not careful we may overlook the fact that it is also an instance of a more complex valid form. Perhaps Russell is biased and has given a very simplistic account of the argument form involved. Perhaps a deeper analysis would reveal that the cosmological argument is also an instance of a more complex form that is in fact valid. Perhaps the argument is valid but not in virtue of form. Perhaps not. The question of the nature of logical form is one to which we will often return. But the question of the logical form of the cosmological argument need not worry us here. It is sufficient to note just how powerful and valuable an ally formal logic can be in debate in natural language, whatever the topic under discussion might be.

In truth, the form which Russell appeals to here is of quite a high level of complexity; as, indeed, is the very first example about eternal happiness and the cheese sandwich which we considered on p. 3. Formal logic can handle forms of this level of complexity with ease and can, in fact, handle still more complex forms of argument. (Such argument forms will be considered in detail later in Chapter 5.) Classical formal logic will prove itself to be an enormously efficient instrument for investigating the nature of argument and the concept of validity itself. To discover precisely how and why that should be the case can be genuinely exciting and will, on occasion, lead to some rather surprising results. Not all of the surprises are pleasant ones, however. Formal logic has its limits.

As we have already seen, for example, there are serious questions about whether the formal logician can ultimately account for validity in purely formal terms. Worse still, perhaps, is the fact that classical formal logic sanctions as valid some forms of argument which are rather less than intuitive. These particular limits will be considered later when we are in a better position to appreciate them for what they are. None the less, the existence of certain possible limitations to the formal logician's project detracts not one iota from the value of studying logic in general and classical formal logic in particular.

Provided that you do not lose sight of the applicability of formal logical considerations to ordinary discourse, you will quickly realise that the study of formal logic tends to produce clear-thinking, articulate individuals who can present and develop complex arguments in a rigorous way. In acquiring these communications skills you will also acquire the ability to lead discussion in a structured way and to persuade others. Further, as we have noted, formal logic provides impressive analytical machinery with which to identify the logical structure of an opponent's arguments and provides an arsenal of weaponry which may well enable one to destroy the apparent force of those arguments. All these skills are obviously valuable and useful to their possessor. Less obviously, perhaps, they are also highly coveted by many employers, particularly in the business environment.

Finally, no logic student should ever lose sight of the enormous practical value of formal logic. In 1879, while Professor of Mathematics at Jena University in Germany, Gottlob Frege [1848–1925] produced the first formal, mathematical language capable of expressing argument-forms as complex as and even more complex than those we have been considering here. The publication of Frege's *Begriffsschrift* is an event whose significance in the development of formal logic is inestimable. The publication of Frege's text certainly heralds the dawn of the modern tradition of classical formal logic with which we are concerned. Moreover, Frege's work not only constituted the first system of modern formal logic but also laid much of the foundations for the contemporary programming languages which have become such an integral part of modern daily life, from the university, college or office software package to automatic cash dispensers and bar tills. The name of the programming language PROLOG, for example, is simply shorthand for *Logic Programming*. Logic is, and always has been, an integral part of philosophy. Students of philosophy in particular should be pleased to be able to lay to rest so easily the old but still popular misconception that their subject is 'impractical' and 'unproductive'!

VIII
A Brief Note on the History of Formal Logic

In all honesty, it will be some time before you become fully aware of the value and extent of Frege's contribution to the development of formal logic. In fact, this will not really become clear until we consider the logic of **general sentences** (sentences involving terms such as 'all' and 'some', 'most' and 'many') and arguments composed of such sentences, again, in Chapter 5 of the present text. The logic of such sentences and arguments is known as **quantificational logic** and the design of the logical machinery of quantificational logic is due, above all, to Gottlob Frege. It is precisely that design which is undoubtedly the crowning glory of Frege's contribution to the development of formal logic and, perhaps, the crowning glory of formal logic itself.

As a first step towards an appreciation of the value of Frege's contribution consider the following historical sketch carefully.

The first system of logic which allowed philosophers to investigate the logic of general sentences formally was designed by Aristotle some 2,000 years before Frege. The importance of Aristotle's own role in the history of formal logic is also unique and inestimable just because formal logic itself originates in the work of that author. As the logician Benson Mates puts it:

> Aristotle, according to all available evidence, created the science of logic absolutely *ex nihilo*.[9]

Moreover, the science which Aristotle created is, as we might put it, properly formal, for it embodies the insight that the validity of certain particular arguments consists in the logical forms which they exemplify. Further, Aristotle's approach to formal logic is generally systematic, i.e. it identifies and groups together the valid forms of argument in an overall system.

Aristotle's system of logic is known as **syllogistic** just because it confines itself to a certain kind of argument known as a *syllogism*. A syllogism consists of two premises and a conclusion each of which is a general or *categorical* *sentence*, i.e. a sentence which makes an assertion about sets of things. Typically, such sentences will tell us that some set is or is not contained in another. So, for example, sentences such as 'All A is B' and 'Some B is C' are categorical sentences. In fact, Aristotle distinguished four different kinds of categorical sentence as foundational in his system of logic. Given a careful analysis of the place and role of the key terms in each such sentence, and of the place of each sentence in a syllogism, no fewer than 256 kinds or *moods* of syllogism can ultimately be distinguished.

As a whole, however, the system of syllogistic is a limited one which does not consider the logic of relations, for example. Moreover, certain elements of the system are, to say the least, controversial. In particular, it is not at all clear that the nature and consequences of general sentences involving 'all' are properly represented. Here we must be very careful. No sentence of the form 'All As are B' ever implies that there actually exist any As. That would be an independent claim. What does follow is just that if there is something which is A *then* that something is also B. To ignore this point can lead to legitimating fallacious reasoning. But it may well be the case that Aristotelian syllogistic does ignore this point. Bertrand Russell points out that just such a fallacy is involved in certain instances of one particular mood of Aristotelian syllogistic. During the Middle Ages the syllogisms were given rather exotic names such as 'Barbara' and 'Celarent' by the medieval logicians who studied them. The syllogism which Russell highlights here is known as 'Darapti' and is of the following form:

. All As are B.

. All As are C.

Therefore,

. Some Bs are C.

As you might expect, Russell offers a counterexample which, this time, concerns a mythical fire-breathing animal, the chimera:

. All chimeras are animals.

. All chimeras breathe flame.

Therefore,

. Some animals breathe flame.

Aristotle's syllogistic was developed and extended by logicians and philosophers throughout the Middle Ages and, indeed, in subsequent centuries. Moreover, the medieval logicians conducted their own logical investigations of general sentences and made considerable progress towards a systematic theory of the logic of such sentences.[10]

The next major step forward in the development of the subject did not occur until well into the nineteenth century when the English logicians George Boole [1815–64] and Augustus De Morgan [1806–71] approached formal logic in terms of abstract algebra and, for the first time, developed algebraic logic. To this day, the algebraic perspective remains a useful and insightful one. In that mathematisation of the subject a new level of formal rigour and systematisation was achieved and, as we shall see in Chapter 4, De Morgan also contributed some extremely useful logical laws. In a sense, these logicians realised the dream of the great German philosopher and logician G.W.Leibniz [1646–1716], who had already outlined the idea of a universal calculus into which arguments could be translated and assessed. But it was not until the work of Frege in the late nineteenth century that the level of systematisation which formal logic now enjoys was achieved. As we shall see, contemporary formal logic is nothing less than a formal language into which arguments can be translated and in which they can be proved to be valid or invalid. Further, Frege not only realised Leibniz's dream but also contributed the machinery of quantificational logic, which enables the logician to dig into even the internal grammatical structure of natural language sentences. So, the particular moment in which we are studying formal logic together is one at which the subject has attained its highest level of achievement in an evolution of more than 2,000 years.

I shall say no more about the historical evolution of formal logic here. Interested parties can explore the development of logic in William and Martha Kneale's useful and accessible text *The Development of Logic* [1962]. A much briefer discussion can also be found in Chapter 12 of Benson Mates's *Elementary Logic* [1972] and, indeed, Mates's *Stoic Logic* [1953] remains a classic in the field. On Aristotelian logic in particular, I also warmly recommend both Jan Lukasiewicz's *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* [1951] and Jonathan Lear's *Aristotle and Logical Theory* [1980].

Before we turn to the first exercise in this text, Exercise 1.1, note carefully that at the end of each chapter I give a summary box of salient points, all of which are generally helpful for the exercises which follow them. (Because certain chapters cover a considerable amount of material you can also expect to find similar summary boxes at regular intervals during the course of relevant chapters.) You might also have noticed that during this chapter key words have been written in bold. Information on these items can be found in the glossary before the index at the end of the text and you will also find entries under these items in the index. Do study the contents of each summary box carefully before attempting subsequent exercises! The first summary box is Box 1.1.

BOX 1.1

- An *argument* consists of premises, a 'therefore'-type word and a conclusion.
- An argument is *valid* if, and only if, it is impossible that its premises be true *and* its conclusion false.
- An argument is *invalid* if, and only if, it *is* possible that its premises be true *and* its conclusion false.
- A *sound* argument is a valid argument with true premises.
- The claim of the formal logician is that an argument is valid purely in virtue of being a *substitution-instance* of a valid *argument form*.
- An argument form is valid if and only if every substitution-instance of that form is valid.
- An argument form is invalid if some substitution-instance of that form is invalid.
- A *counterexample* to a form is a substitution-instance of that form which is itself an invalid argument.

EXERCISE 1.1

1 For logical purposes, an argument is a set of sentences in which some sentences (the premises) purport to give reasons for accepting another sentence (the conclusion). But not every set of sentences constitutes an argument. Consider the following sets of sentences carefully. In each case, state whether or not that set of sentences constitutes an argument. Give reasons for your answers.

A Professor Plum was in the drawing room. Miss Scarlet was in the kitchen. The murderer used the knife and the evil act was committed in the hall.

B If Professor Plum was in the drawing room then Colonel Mustard was the murderer. Professor Plum was in the drawing room. So, Colonel Mustard was the murderer.

C Every student of logic is wise and knowledgeable. Anyone attempting this exercise is a student of logic. Therefore, anyone attempting this exercise is wise and knowledgeable.

D I am absolutely sick and tired of getting wet every time it rains. From now on I will never forget to take my umbrella with me in the morning. Even if the weather looks fine when I leave I shall certainly make a point of taking that umbrella.

2 In an argument the premises purport to give reasons for accepting the conclusion. In general, the examples considered in this chapter have listed the premises before stating the conclusion. In ordinary discourse, however, premises are not always stated before their conclusions. Study the following arguments carefully. In each case, state which sentences you consider to be premises and which the conclusion. Give reasons for your answers.

A Professor Plum was in the drawing room and Miss Scarlet was in the conservatory. If Professor Plum was in the drawing room and the murder weapon was found in the drawing room then Professor Plum is in big trouble. So, if the murder weapon was found in the drawing room then Professor Plum really is in big trouble.

B All human beings are mortal. So, it stands to reason that Socrates is mortal. After all, he is a human being.

C Very few elephants can fly. Very few elephants are pink. So, the pink flying elephant is truly a rare creature for fewer pink elephants than ordinary elephants can actually fly.

D Professor Plum was obviously the murderer in this instance. For the murderer used the knife and Professor Plum had the knife. And the murder was committed in the hall and Professor Plum was certainly in the hall earlier.

3 An argument is valid if and only if it is impossible that its premises be true and its conclusion false. Consider the following questions carefully before responding. In each case give reasons for answering as you do.

A When is an argument invalid?

B Can a valid argument have a false conclusion?

C Can a valid argument have actually true premises but a false conclusion?

D Can an argument have true premises and a true conclusion but not be valid?

E Can an argument be sound but invalid?

F Must the conclusion of a sound argument be true?

G When is an argument-form valid?

H When is an argument-form invalid?

4 Consider the following arguments carefully. In each case, indicate whether the argument is valid or invalid. If you find any to be valid indicate whether or not the argument is also sound:

A 1. If Abraham Lincoln was French then the Moon is made of green cheese.

2. Abraham Lincoln was French.

Therefore,

3. The moon is made of green cheese.

B 1. The Washington Redskins are better than the Miami Dolphins.

2. But the Miami Dolphins are better than the Buffalo Bills.

Therefore,

3. The Washington Redskins are better than the Buffalo Bills.

C 1. If all cats are black then Tiffin is black.

2. Tiffin is black.

Therefore,

3. All cats are black.

D 1. If all cats are black then Zebedee is black.

2. Some cats are not black.

Therefore,

3. Zebedee is not black.

5 (i) Using only: 'If…then—;', '*p*', '*q*', and 'not' exhibit the logical form of arguments C and D in 4 above.

(ii) State whether or not my natural language arguments C and D constitute counterexamples to those logical forms. Give reasons for your answers.

6 Provide a counterexample to the following argument-form:

1. If *p* then not *q*

2. Not *p*

Therefore,

3. Not *q*

7 Using only sentential variables exhibit the logical form of the following argument:

1. The team strip is red.

Therefore,

2. The team strip is coloured.

State: (i) whether the form is valid or not and (ii) whether the particular argument itself is valid or not.

For discussion:
What, in your view, do your answers imply as regards the purely formal definition of validity?

Notes

1 I am indebted to John Slaney for the kind of example involved here.
2 Lyndon, Roger C., [1966], *Notes on Logic,* Van Nostrand Mathematical Studies #6, Princeton NJ, D.Van Nostrand, preface, p. iii.
3 See, for example, Popper, K.R., [1972], *Conjectures and Refutations,* fourth edition, London and Henley, Routledge & Kegan Paul, Ch. 3.
4 In Phillips, Calbert (ed.), [1995], *Logic in Medicine,* London, British Medical Journal Publishing Group, Ch. 2. But see also Ch. 1 of this volume.
5 Quine, W.V.O., [1986], *Philosophy of Logic,* second edition, Cambridge MA and London, Harvard University Press.
6 I am indebted to Stephen Read for this point which was made in correspondence. However, the point is also well made in Read, Stephen, [1995], *Thinking about Logic: An Introduction to the Philosophy of Logic,* Oxford, Oxford University Press, Ch. 2. Interested parties will find a substantive discussion of relevant issues there and a useful guide to further reading in the area.
7 Wittgenstein, Ludwig, [1967], *Philosophical Investigations,* Oxford, Blackwell, Remark 81.
8 Luce, A.A., [1958], *Logic,* London, English Universities Press, p. 9.
9 Mates, Benson, [1972], *Elementary Logic,* second edition, New York, Oxford University Press, p. 206.
10 We cannot pursue this fascinating aspect of the development of formal logic but interested parties might profitably consult Broadie, Alexander, [1987], *Introduction to Medieval Logic,* Oxford, Clarendon Press. See also Boehner, Philotheus, [1952], *Medieval Logic: An Outline of its Development from 1250 to c.1400,* Manchester, Manchester University Press.