LOGICISM AND ITS PHILOSOPHICAL LEGACY

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CAMBRIDGE
UNIVERSITY PRESS

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Introduction

The idea that mathematics is reducible to logic has a long history. But it was Frege who gave logicism an articulation and defense that transformed it into a distinctive philosophical thesis of major consequence. With Frege's intervention, the doctrine came to have a profound influence on the development of philosophy in the twentieth century: it led directly to modern second-order logic – essentially the product of Frege's analysis of the concept of generality – and to notions of analysis that came to define philosophical thinking in Britain, on the continent, and in America.

The recent revival of interest in Frege's philosophy of arithmetic has shown it to be of much more than just historical interest. This revival was occasioned by Crispin Wright's important discovery that a great deal of Frege's theory of number is unaffected by the contradiction in his theory of classes. The reevaluation of Frege inspired by Wright's discovery has spawned a large philosophical and technical literature. Though informed by the technical results recorded in this literature, the goal of the essays collected here is general and philosophical. In addition to their contribution to the reevaluation of Frege's philosophy of mathematics, they argue that Frege's work contains insights that bear on fundamental problems of interpretation that arise in the context of empirical theories. The essays offer a series of critical reflections on a variety of applications of different conceptions of analysis – all of them having their source in the logicist tradition – to the philosophy of mathematics and the philosophy of science. An assumption they attempt to encourage is that there is a great deal still to be learned about the nature of analysis and how it was practiced by Frege and the principal historical figures in the logicist tradition who followed him – Russell, Ramsey, and Carnap.

While I hope that the essays in this volume will be of interest to historians, especially historians of the twentieth-century analytic tradition, they are not – nor do they pretend to be – scholarly historical studies. I have tried to exhibit the main features of the variety of alternative analytical approaches that different logicists have pursued in their separate attempts to define what is distinctive about arithmetical knowledge. And I have been especially concerned to isolate what was characteristic of Frege's contribution to the analysis of our knowledge of number, and to illuminate, by comparison with his contribution, the analyses of others. But I have not hesitated to emphasize a reconstruction of an author's view if I thought doing so was clarifying or suggestive of a more successful approach to the problem I was considering; I have proceeded in this way even when such a reconstruction represented an obvious departure from the view that inspired it. This is particularly true of my accounts of Frege's analysis of arithmetical knowledge in Chapter 1, Carnap's elaboration and defense of the thesis that mathematics is not factual in Chapter 2, and *Principia*'s theory of functions and classes in Chapter 10.

Frege's criterion of identity for number occupies a prominent position in these essays. The criterion was put forward in *Grundlagen* as an "informative" answer to the question, 'Under what conditions should we regard statements of the form, "The number of *F*s is the same as the number of *G*s," true?' According to Frege, such statements express "recognition judgments": they express our recognition that the same number has been presented in two different ways, as the number of one or another concept. The criterion of identity asserts that a recognition judgment is true if and only if its constituent concepts are in one–one correspondence.

In the secondary literature, Frege's criterion of identity is sometimes referred to as a partial contextual definition of the truth conditions for numerical statements of identity. Setting aside the notion of a contextual definition, the definition the criterion expresses is said to be *partial* because it covers only some identities involving numerical singular terms. It covers those cases where the singular terms are of the form 'the number of . . .,' or, as with numeral names, are definable in terms of expressions of this form, but it leaves out cases where at least one of the singular terms is not of this character. However, even if the criterion of identity is in this sense only a partial contextual definition, when taken as an axiom or principle, it suffices for a mathematically adequate theory of number – one that gives rise to an analysis of arithmetical knowledge of considerable philosophical interest.

In the essays that follow, Frege's criterion of identity is discussed in connection with two rather different applications of it. It is discussed first in Chapter 1 in connection with the theory of arithmetical knowledge, where I formulate a theory that is *close* to Frege's intended account and compare it with the theory of Whitehead and Russell. In Chapter 2 the criterion is discussed in connection with Carnap, where it is applied to the problem of distinguishing between formal and factual components of the language of science. The essays of both chapters are new to this volume, but since the second application of Frege's criterion of identity is likely to be a much less familiar kind of application of the criterion than the first, it might be worthwhile to comment further on it here.

Hume had the idea that the difference between the epistemic status of arithmetic and geometry can be traced to the different criteria of identity that govern their fundamental notions – the notions of number and length, respectively. Indeed, when in sect. 63 of *Grundlagen*, Frege introduces the criterion of identity, he quotes from Hume's *Treatise*:

> We are possest of a precise standard, by which we can judge of the equality and proportion of two numbers; and according as they correspond or not to that standard, we determine their relations without any possibility of error. When two numbers are so combin'd as that one has always an unite answering to every unite of the other, we pronounce them equal[.] (Book I, Part III, sect. I, para. 5)

That Frege quotes Hume in the course of introducing his criterion of identity for number led George Boolos to coin the name 'Hume's principle' for the criterion of identity. Motivated in no small measure by the attractiveness of Boolos's essays on the topic, the name has become established in the secondary literature devoted to the reevaluation and revival of Frege's contributions to the philosophy of mathematics. Whether or not it was intended, Boolos's terminology suggests an important change in perspective: it focuses attention on the role of the criterion of identity as an independently motivated axiom, rather than a definition that is incomplete or partial.

It is interesting to note that Frege does not quote Hume in full, but leaves out the remark with which the passage concludes:

> and 'tis for want of such a standard of equality in extension, that geometry can scarce be esteem'd a perfect and infallible science.

Frege had little interest in Hume's application of criteria of identity – "standards of equality" in Hume's phrase – to the problem of elucidating the epistemological differences between arithmetic and geometry. But he certainly hoped to gain acceptance for Hume's principle, given its pivotal position in the central mathematical argument of *Grundlagen*: the entire development of the theory of number is based on it, and Frege may well have supposed that by citing the precedent of Hume, he could enhance the plausibility of basing his theory on this criterion. Nevertheless, Frege's favorable citation of Hume obscures important differences in their understanding of the criterion of identity: it is clear from the passage Frege quotes that Hume conflates the notion of a class with that of a plurality, and that he treats a plurality of things as a "number," rather than treating numbers as separate from – and representative of – the concepts or classes to which they "belong." Each of these differences points to a characteristic that is a hallmark of Frege's philosophy of mathematics.

The context of Hume's introduction of the standard of equality for number is actually *geometric*, and arithmetic enters the discussion only coincidentally. Hume observed that the idea that space is infinitely divisible cannot be based on perception or imagination since reflection quickly reveals that any division we effect must terminate after some indefinite but finite number of steps. The terminus of any process of division is not susceptible to further division, and although the termini are in this sense unextended spatial minima, they are minima that are reached after finitely many divisions. Against the suggestion that two spatial intervals are equal in length if they are composed of the same number of spatial minima, Hume argues that our idea of an interval is always of one for which the minima are "confounded" with one another. This confounding has the consequence that it is impossible to base a notion of sameness of length on a one–one correspondence between the minima belonging to the two intervals. The situation is altogether different in the case of our judgments of sameness of number, for these concern pluralities of "unites" which we can clearly grasp as separate from one another. Hence the question whether the units of the pluralities to which arithmetic is applied are or are not in one–one correspondence has a clear sense and a determinate answer. But in the case of geometry, we must be content with the usual criterion for sameness of length, namely the coincidence of the end-points of the intervals being compared: in other words, the geometrical situation is just as it would be if we were dealing with a continuum, where comparisons are necessarily approximate. Geometry is therefore fallible and "imperfect" in a way that arithmetic is not.[1]

In Chapter 2 I argue that there is a sense in which the question, 'What distinguishes applied geometry from applied arithmetic?' is illuminated by an examination of the criteria of identity that are appropriate to arithmetical and geometric notions. Although Hume must be credited for first raising

[1] Notice that the appropriateness of the arithmetical criterion in the context of a discrete space can be reasonably questioned even if one does not accept the relevance of Hume's contention that spatial minima are necessarily "confounded" by us. It would not be unreasonable to reject the arithmetical criterion for sameness of length if the judgments of congruence to which it led conflicted with those that are forced by our customary geometric procedures for establishing congruence.

the possibility that differences associated with their criteria of identity bear on the difference in epistemic status we attribute to applied geometry and applied arithmetic, his positive proposals fail to isolate the correct point. Contrary to Hume, the difference is not that one standard is precise in a way that the other can never be, or that one enjoys a kind of infallibility that the other can never attain. Rather the central difference is that the criteria of identity for geometrical notions like *length* are empirically constrained in a way that the criterion of identity for *number* is not. The manner in which geometrical notions are differently constrained depends on methodological considerations of some subtlety, requiring for their full appreciation the discussion of chronometry and the criterion of identity for *time of occurrence*. The chapter concludes by showing how these observations can be exploited to support Carnap's claim that applied geometry has a factual content that applied arithmetic lacks. This claim lies at the center of Carnap's insistence that an adequate reconstruction of physics must include a principled distinction between its factual and nonfactual components.

There is an interesting conceptual difference between Frege and Hume which would have been clear to Frege, but would have fallen outside Hume's conceptual horizon. This difference is attributable to their divergent views on the scope of logic. Although a small point, it leads naturally to an observation about Russell and a major preoccupation of several of the essays collected here. Hume's discussion of infinite divisibility raises two questions: (i) 'What is the basis for our belief that space is infinitely divisible?' and (ii) 'How do we come to the concept of infinite divisibility?' Hume treats the first question as one about space as we perceive or imagine it; he then proceeds to address the question on the basis of the phenomenological evidence that is provided by our perception and imagination. As for the second question, Frege had the conceptual resources from which he could argue that whether or not the basis for our *belief* in infinite divisibility is decided by what is allowed by our perceptual and imaginative capacities, our *concept* of infinite divisibility does not rest on perception or imagination *because infinite divisibility – i.e. denseness – is a purely logical notion*. Even if Hume were to agree with the conclusion that our concept of infinite divisibility does not rest on our perception or imagination, he could not have advanced this argument on its behalf.

Similarly, independently of whether the basic laws of arithmetic are reducible to the laws of logic, it was a discovery of some interest that the truth of a recognition judgment should depend on a condition that is characterizable in terms drawn exclusively from logic. That this notion of similarity of properties or Fregean concepts is a wholly logical one, and therefore a notion of similarity that does not depend on experience or "a priori intuition," was the insight that led Russell to the view that the notion of *structural similarity* – which is a simple extension of one–one correspondence – might be capable of resolving classical metaphysical and epistemological problems about the relation of appearance to reality. The origins of Russell's "structuralism" therefore trace back to ideas that were integral to the development of logicism. Chapter 4 contains an extended introduction to the essays that deal with Russell's structuralism and its elaboration in the work of Ramsey, Carnap, and others, and its initial sections should be consulted for an overview of the topics these essays address.

Although the order of the chapters is the order in which I believe the essays are best read, each essay stands by itself and can be read independently of the others. I should, however, note that the previously unpublished essays were written not only to advance new ideas, but to introduce and orient the reader to the older essays in this collection. Indeed, the first four essays have a particularly strong claim to being read before the others and in the order in which they are presented, and they come close to forming a monograph of their own. As I mentioned earlier, Chapter 1 refines my view of what is of contemporary interest in Frege's theory of number and presents the background to Chapter 2; Chapter 1 also serves as an introduction to all the essays that are specifically concerned with logicism and the philosophy of mathematics, namely Chapters 8–11. Chapter 2 is the chapter most explicitly concerned with applying the central ideas of Frege's logicism to issues in the philosophy of science, and it forms a natural transition from Chapter 1 to Chapters 3 and 4. Chapter 3 addresses whether someone sympathetic to Carnap's views on questions of realism in the philosophy of mathematics is committed to a similar view of such questions in the philosophy of physics. In particular, it considers whether his distinction between internal and external questions can be applied differentially to questions about the existence of abstract entities and questions about the existence of the theoretical entities of physics. Chapter 4 is not only an introduction to Chapters 5–7; it also connects the discussion of Carnap's distinction between internal and external questions with issues raised by "structuralist" theories of theoretical knowledge, and it complements and extends the earlier discussion in Chapter 3 of the reality of theoretical entities.

In general, I have made only stylistic changes or changes for the purpose of clarification to the previously published essays. The major exception is Chapter 10 which, in addition to many such changes, has been significantly expanded by the addition of an appendix on Russell's propositional paradox, its extension to Fregean thoughts, and its relevance to Dedekind's theory of our knowledge of number. Where I have found a formulation or point of view with which I no longer agree, I have, in cases where it seemed important to do so, indicated this by a new footnote; but I have refrained from rewriting passages to reflect my current view. All footnotes that record such disagreements or direct the reader to a subsequent discussion use a special mark and are annotated as having been added in 2012.

CHAPTER 1

Frege's analysis of arithmetical knowledge

> Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object thus constructed.
>
> (Kant 1787, A716/B744)

Frege most directly engages Kant when, in *Grundlagen*,[1] he presents his formulations of the analytic–synthetic and a priori–a posteriori distinctions. The discussion of Frege's account of these distinctions has usually focused on whether the conception of logic which informs his definition of analyticity undermines his claim to have shown that arithmetic is analytic in a sense that Kant would have been concerned to deny. I will argue that Frege has an elegant explanation of the *apriority* of arithmetic, one that challenges Kant even if Frege's claim to have reduced arithmetic to logic is rejected. In the course of discussing Frege's explanation of the apriority of arithmetic, I will also clarify certain fundamental differences between his and Whitehead and Russell's theories of number – differences which bear importantly on their respective accounts of the nature of arithmetical knowledge. I will show that even if Frege's understanding of Kant was defective in every detail, knowing what he took himself to be reacting against and correcting in Kant's philosophy of arithmetic is of interest for the light it casts on the development of logicism.

1 Frege (1884). My citations of Frege (1879), (1884), and (1893/1903) – his *Begriffsschrift*, *Grundlagen*, and *Grundgesetze*, respectively – use the mnemonic abbreviations, *Bg*, *Gl*, and *Gg*, followed by the relevant page or section number.

1.1 FREGE'S INTEREST IN RIGOR

It may seem obvious that Frege's interest in rigor – his interest in providing a framework in which it would be possible to cast proofs in a canonical gap-free form – is driven by the problem of providing a proper justification for believing the truth of the propositions of arithmetic. One can easily find quotations from Frege which show that he sometimes at least wrote as if he took his task to be one of securing arithmetic; and it would be foolish to deny that the goals of cogency and consistency were an important part of the nineteenth-century mathematical tradition of which Frege was a part. Nevertheless, I think it can be questioned how far worries about the consistency or cogency of mathematics, generated perhaps by a certain incompleteness of its arguments, were motivating factors for Frege's logicism or for the other foundational investigations of the period.

There is another, largely neglected, component to Frege's concern with rigor that not only has an intrinsic interest, but also elucidates his views on the nature and significance of intuition in mathematical proof and his conception of his mathematical and foundational accomplishments. According to this component, Frege's concern with rigor is predominantly motivated by his desire to show that arithmetic does not depend on Kantian intuition, a concern Frege inherited from the tradition in analysis initiated by Cauchy and Bolzano, and carried forward by Weierstrass, Cantor, and Dedekind.

Shortly after the publication of *Begriffsschrift* Frege wrote a long study of its relationship to Boole's logical calculus.[2] The paper carries out a detailed proof (in the notation of *Begriffsschrift*) of the theorem that the sum of two multiples of a number is a multiple of that number. In addition to the laws and rules of inference of *Begriffsschrift*, Frege appeals only to the associativity of addition and to the fact that zero is a right identity with respect to addition. He avoids the use of mathematical induction by applying his definition of *following in a sequence* to the case of the number series. The paper also includes definitions of a number of elementary concepts of analysis (again in the notation of *Begriffsschrift*). It has been insufficiently emphasized that neither in this paper nor in *Begriffsschrift* does Frege suggest that the arithmetical theorems proved there are not correctly regarded as self-evident, or that without a *Begriffsschrift*-style proof, they and the propositions that depend on them might reasonably be doubted. Frege's point is rather that without gap-free proofs one might be misled into

[2] Frege (1880/1), published only posthumously.

thinking that arithmetical reasoning is based on intuition. As he puts the matter in the introductory paragraph to Part III of *Begriffsschrift*:

> Throughout the present [study] we see how pure thought, irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgements that at first sight appear to be possible only on the basis of some intuition.

The same point is made in *Grundlagen* when, near the end of the work (sects. 90–1), Frege comments on *Begriffsschrift*. Frege is quite clear that the difficulty with gaps or jumps in the usual proofs of arithmetical propositions is not that they might hide an unwarranted or possibly false inference, but that their presence obscures the true character of the reasoning:

> In proofs as we know them, progress is by jumps, which is why the variety of types of inference in mathematics appears to be so excessively rich; the correctness of such a transition is immediately self-evident to us; whereupon, since it does not obviously conform to any of the recognized types of logical inference, we are prepared to accept its self-evidence forthwith as intuitive, and the conclusion itself as a synthetic truth – and this even when obviously it holds good of much more than merely what can be intuited.
>
> On these lines what is synthetic and based on intuition cannot be sharply separated from what is analytic . . .
>
> To minimize these drawbacks, I invented my concept writing. It is designed to produce expressions which are shorter and easier to take in . . . so that no step is permitted which does not conform to the rules which are laid down once and for all. It is impossible, therefore, for any premiss to creep into a proof without being noticed. In this way I have, without borrowing any axiom from intuition, given a proof of a proposition[3] which might at first sight be taken for synthetic.

It might seem that to engage such passages it is necessary to enter into a detailed investigation of the Kantian concept of an a priori intuition. But to understand Frege's thought it is sufficient to recall that for the Kantian mathematical tradition of the period our a priori intuitions are of space and time, and the study of space and time falls within the provinces of geometry and kinematics. It follows that the dependence of a basic principle of arithmetic on some a priori intuition would imply that arithmetic lacks the autonomy and generality we associate with it. To establish its basic principles, we would have to appeal to our knowledge of space and time; and then arithmetical principles, like those expressing mathematical induction and various structural properties of the ancestral, would ultimately

[3] The proposition to which Frege refers is the last proposition proved in *Begriffsschrift* – Proposition 133 – which states that the ancestral of a many–one relation satisfies a restricted form of connectedness.

come to depend for their full justification on geometry and kinematics. Frege's point in the passages quoted is that even if it were possible to justify arithmetical principles in this way, it would be a mistake to suppose that such an external justification is either necessary or appropriate when a justification that is internal to arithmetic is available. From this perspective, the search for proofs, characteristic of a mathematical investigation into foundations of the sort Frege was engaged in, is not motivated by any uncertainty concerning basic principles and their justification, but by the absence of an argument which establishes their autonomy from geometrical and kinematical ideas. And autonomy is important since it is closely linked to the question of the generality of the principles of arithmetic.

On this reading of the allusion to Kantian intuition, the rigor of *Begriffsschrift* is required in order to show that arithmetic has no need of spatial or temporal notions. Indeed, Frege's concern with autonomy and independence suffices to explain the whole of his interest in combating the incursion of Kantian intuition into arithmetic. In this respect, Frege's intellectual motivation echoes that of the nineteenth-century analysts. Thus as early as 1817 Bolzano wrote: "the concepts of *time* and *motion* are just as foreign to general mathematics as the concept of *space*."[4] And over fifty years later Dedekind was equally emphatic:

> For our immediate purpose, however, another property of the system $\mathfrak{R}$ [of real numbers] is still more important; it may be expressed by saying that the system $\mathfrak{R}$ forms a well-arranged domain of one dimension extending to infinity on two opposite sides. What is meant by this is sufficiently indicated by my use of expressions borrowed from geometric ideas; but just for this reason it will be necessary to bring out clearly the corresponding purely arithmetic properties in order to avoid even the appearance [that] arithmetic [is] in need of ideas foreign to it.[5]

It must be conceded that the issues raised by the foundations of analysis are more varied than those that arise in the case of the arithmetic of the natural numbers; in the case of the real numbers, the usual explanation of the purpose of rigor – that it is a guarantee of cogency and a hedge against inconsistency and incoherence – while not a complete account of the matter, is certainly a significant part of the story. But when we turn to Frege's primary concern about the domain of intuition – namely, the arithmetic of natural numbers – skepticism simply plays no role in any of his arguments against its use: Frege nowhere rejects intuition for fear that it is a potentially faulty guide to truth. Those few passages which suggest

[4] Russ (1980, p. 161). [5] Dedekind (1872, p. 5).

otherwise are invariably concerned with arithmetic in the broad sense, which includes real analysis. When we factor arithmetic by real analysis we also factor out doubts about cogency and consistency. What is left is a concern with autonomy and independence of the sort expressed by Bolzano and Dedekind in the passages just quoted.

We may briefly summarize our discussion of the historical situation as follows. The interest in rigor has both philosophical and mathematical aspects. To begin with, we require proofs internal to arithmetic. This idea is invariably presented as a prohibition against the incursion of spatial and temporal notions into demonstrations of the propositions of arithmetic. Finding proofs which are free of such notions is the mathematical aspect of rigorization. But the motivation underlying the foundational interest in rigor also has an important philosophical component which is broadly architectonic: the philosophical dimension to Frege's foundational program is to establish the independence of our knowledge of arithmetical principles from our knowledge of spatial and temporal notions.

1.2 THE SIGNIFICANCE OF A DERIVATION OF ARITHMETIC FROM LOGIC

Anyone familiar with the thesis that arithmetic is part of logic but with only a passing acquaintance with Frege's writings might expect that Frege was concerned to show that the necessity of arithmetic is inherited from the necessity of the laws of logic. But there are two considerations that argue against this expectation. First, Frege barely addresses the question of what characterizes a truth as logical, and when he does, necessity plays no role in his answer. Frege's great contribution to logic was his formulation of higher-order polyadic logic with mixed generality; but he contributed very little to our understanding of what constitutes a logical notion or a logical proposition and still less to our understanding of logical necessity. For Frege the laws of logic are distinguished by their universal applicability. Secondly, in so far as the laws of logic are for Frege necessary, they are "necessities of thought," a feature that is exhibited by the fact that they are presupposed by all the special sciences. But for Frege, this is a feature that is shared by the laws of arithmetic – whether or not they are derivable from logic. Because of logic's universality, a reduction of arithmetic to it would preserve the generality of arithmetic, but given Frege's conception of logic, it would shed no light on either arithmetic's necessity or its generality, since it is clear that these are properties arithmetic already enjoys.

The necessity of arithmetic is implicit in Frege's acknowledgment that it is a species of a priori knowledge. On Frege's definition of 'a priori' this means that arithmetic is susceptible of a justification solely on the basis of general laws that neither need nor admit of proof.[6] To say that a law neither needs nor admits of proof is to say that the law is warranted, and no premise of any purported proof of it is more warranted than the law itself. To the extent to which this implicitly assumes a notion of necessity, it is a wholly epistemic one. But while Frege's understanding of necessity is entirely epistemic, the contemporary concern with the necessity of arithmetic is characteristically directed toward the metaphysical necessity of arithmetical truths; this concern is altogether absent in Frege, and became prominent in the logicist tradition only much later, with Ramsey's celebrated 1925 essay on the foundations of mathematics.[7]

Frege's understanding of the epistemological significance of a derivation of arithmetic from logic is subtle. Certainly such a derivation would make it clear that arithmetic is not *synthetic* a priori, which is something Frege certainly sought to establish. But the derivation of arithmetic from logic is not needed for the simpler thesis that arithmetical knowledge is encompassed by Frege's definition of a priori knowledge. To suppose otherwise would be to imply that an appeal to logical principles is required because there are no self-warranting *arithmetical* principles to sustain arithmetic's apriority. This could only be maintained if arithmetical principles were less warranted than the basic laws of logic, or if they lacked the requisite generality. Discounting, for the moment, the second alternative, but supposing arithmetical principles to be less warranted than those of logic, the point of a derivation of arithmetic from logic would be to provide a justification for arithmetic. But Frege did not maintain that the basic laws of arithmetic – by which I mean the second-order Peano axioms (PA^2) – are significantly less warranted than those of his logic.[8]

It is important always to bear in mind that *Grundlagen* was not written in response to a "crisis" in the foundations of mathematics. *Grundlagen* seeks above all to illuminate the character of our knowledge of arithmetic and to

[6] Sect. 3 of *Grundlagen* appears to offer this only as a sufficient condition, not as a necessary and sufficient condition as a definition would require. Here I follow Burge (2005, pp. 359–60) who argues, convincingly in my view, that the condition is intended to be both necessary and sufficient.

[7] See Ramsey (1925b, sect. 1). Ramsey's essay is discussed at length in Chapter 11, below.

[8] I do not regard the equation of the basic laws of arithmetic with the second-order Peano axioms as at all tendentious. Each of the familiar Peano axioms or a very close analogue of it occurs in the course of the mathematical discussion of sects. 74–83 of *Grundlagen*, where the implicit logical context is the second-order logic of *Begriffsschrift* which, as customarily interpreted, assumes full comprehension. For additional considerations in favor of this equation, see Dummett (1991a, pp. 12–13).

address various misconceptions, most notably the Kantian misconception that arithmetic rests on intuitions given a priori. Frege says on more than one occasion that the primary goal of his logicism is not to secure arithmetic, but to expose the proper dependence relations of its truths on others. The early sections of *Grundlagen* are quite explicit in framing this general epistemological project, as the following passages illustrate:

> The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths on one another. (*Gl*, sect. 2)
>
> [T]he fundamental propositions of arithmetic should be proved . . . with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends. (*Gl*, sect. 4)
>
> [I]t is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic. (*Gl*, sect. 4)

Although a demonstration of the analyticity of arithmetic would indeed show that the basic laws of arithmetic can be justified by those of logic, the principal interest of such a derivation is what it would reveal regarding the dependence of arithmetical principles on logical laws. This would be a result of broadly epistemological interest, but its importance would not necessarily be that of providing a warrant where one is otherwise lacking or insufficient.

As for the idea that numbers are abstract objects, this also plays a relatively minor role in *Grundlagen*, and it is certainly not part of a desideratum by which to gauge a theory of arithmetical knowledge. Frege's emphasis is rarely on the positive claim that numbers are abstract objects, but is almost always a negative one to the effect that numbers are not ideas, not collections of units, not physical aggregates, not symbols, and most importantly, are neither founded on Kantian intuition nor the objects of Kantian intuition.

Aside from the thesis that numbers are objects – arguments to concepts of first level – the only positive claim of Frege's regarding the nature of numbers is that they are extensions of concepts, a claim which in *Grundlagen* has the character of a convenience (*Gl*, sects. 69 and 107); nor does he pause to explain extensions of concepts, choosing instead to assume that the notion is generally understood (footnote 1 to sect. 68). Frege's mature, post-*Grundlagen*, view of the characterization of numbers as classes or extensions of concepts is decidedly less casual. But that it too is almost exclusively focused on the epistemological role of classes – they facilitate the thesis that arithmetic is recoverable by analysis from our knowledge of logic – is clearly expressed in his correspondence with Russell:

> I myself was long reluctant to recognize . . . classes, but saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? And I found no other answer to it than this, We apprehend them as extensions of concepts . . . I have always been aware that there are difficulties connected with [classes]: but what other way is there?[9]

We should distinguish two roles that sound or truth-preserving derivations are capable of playing in a foundational investigation. Let us call derivations that play the first of these roles *proofs* (or *demonstrations*), and let us distinguish them from derivations that support *analyses*. *Proofs* are derivations which enhance the justification of what they establish by deriving them from more securely established truths in accordance with logically sound principles. *Analyses* are derivations which are advanced in order to clarify the logical dependency relations among propositions and concepts. The derivations involved in analyses do not enhance the warrant of the conclusion drawn, but display its basis in other truths. Given this distinction, the derivation of arithmetic from logic would not be advanced as a *proof* of arithmetic's basic laws if these laws were regarded as established, but in support of an *analysis* of them and their constituent concepts – most importantly, the concept of number. The general laws, on the basis of which a proposition is shown to be a priori, neither need nor admit of proof in the sense of 'proof' just explained. And although general laws may stand in need of an analysis, the derivation their analysis rests upon may well be one that does not add to their epistemic warrant. Unless this distinction or some equivalent of it is acknowledged, it is difficult to maintain that the basic laws of arithmetic neither need nor admit of proof while advancing the thesis that they are derivable from the basic laws of logic. There may be difficulties associated with the notion of neither needing nor admitting of proof, but they are not those of excluding the very possibility of logicism by conceding that arithmetic has basic laws that neither need nor admit of proof.

1.3 THE PROBLEM OF APRIORITY

Suppose we put to one side the matter of arithmetic's being analytic or synthetic. Does there remain a serious question concerning the mere apriority of its basic laws? This is actually a somewhat more delicate

[9] Frege to Russell, July 28, 1902, in McGuinness (1980, pp. 140–1).

question than our discussion so far would suggest. Recall that truths are, for Frege, a priori if they possess a justification exclusively on the basis of *general* laws which themselves neither need nor admit of proof. An unusual feature of this definition is that it makes no reference to experience. A possible explanation for this aspect of Frege's formulation is his adherence to the first of the three "fundamental principles" announced in the introduction to *Grundlagen*: always to separate sharply the psychological from the logical, the subjective from the objective. Standard explanations of apriority in terms of independence from experience have the potential for introducing just such a confusion – which is why apriority and innateness became so entangled in the traditional debate between rationalists and empiricists. Frege seems to have regarded Mill's views as the result of precisely the confusion that a definition in terms of general laws – rather than facts of experience – is intended to avoid. He therefore put forward a formulation which avoids even the appearance of raising a psychological question.

Although Frege's definition is in this respect nonstandard, it is easy to see that it subsumes the standard definition which requires of an a priori truth that it have a justification that is independent of experience: The justification of a fact of experience must ultimately rest on instances, either of the fact appealed to or of another adduced in support of it. But an appeal to instances requires mention of particular objects. Hence if a justification appeals to a fact of experience, it must appeal to a statement that concerns a specific object and that is therefore not a general law. Therefore, if a truth fails the usual test of apriority, it will fail Frege's test as well. However the converse is not true: a truth may fail Frege's test because its justification involves mention of a particular object, but there is nothing in Frege's definition to require that this object must be an object of experience. This inequivalence of Frege's definition with the standard one would point to a defect if it somehow precluded a positive answer to the question of the apriority of arithmetic. Does it?

Thus far I have only argued that for Frege the basic laws of arithmetic are not significantly less warranted than those of his logic. But are they completely general? In a provocative and historically rich paper, Tyler Burge (2005, sect. v) argues that Frege's account of apriority prevents him from counting all the Peano axioms as a priori. Let us set mathematical induction aside for the moment. Then the remaining axioms characterize the concept of a natural number as Dedekind-infinite, i.e. as in one-to-one correspondence with one of its proper subconcepts. And among these axioms, there is one in particular which, as Burge observes, fails the test of generality because it expresses a thought involving a particular object:

> It is, of course, central to Frege's logicist project that truths about the numbers – which Frege certainly regarded as particular, determinate, formal objects (e.g. *Gl*, sects. 13 and 18) – are derivative from general logical truths . . . [But s]uppose Frege is mistaken, and arithmetic is not derivable in an epistemically fruitful way from purely general truths. Suppose that arithmetic has the form it appears to have – a form that includes primitive singular intentional contents or propositions. For example, in the Peano axiomatization, arithmetic seems primitively to involve the thought that 0 is a number . . . If some such knowledge is primitive – underived from general principles – then it counts as a posteriori on Frege's characterization. This would surely be a defect of the characterization. (Burge 2005, pp. 384–5)

Burge argues that it matters little that for Frege zero is to be recovered as a "logical object," and that for this reason it is arguably not in the same category as the particular objects the definition of a priori knowledge is intended to exclude. The problem is that such a defense of Frege makes his argument for the apriority of arithmetic depend on his inconsistent theory of extensions. Hence the notion of a logical object can take us no closer to an account of this fundamental fact – the fact of apriority – concerning our knowledge of arithmetic. Thus, Burge concludes, unless logicism can be sustained, Frege is without an account of the apriority of arithmetic.

There is, however, another way of seeing how the different components of *Grundlagen* fit together, one that yields a complete solution to what I will call *the problem of apriority*:

> To explain the apriority of arithmetic in Frege's terms, i.e. in terms of an epistemically fruitful derivation from general laws which do not depend on the doctrine of logical objects or the truth of logicism.

The reconstruction of *Grundlagen* that suggests itself as a solution to this problem is so natural that it is surprising that it has not been proposed before. It is, however, a reconstruction, not an interpretation of Frege's views. Frege may have conceived of *Grundlagen* in the way I am about to explain, but there are at least two considerations that argue against such a supposition. First, Frege's main purpose in *Grundlagen* is to establish the analyticity of arithmetic; but on the proposed reconstruction, the goal of establishing arithmetic's apriority receives the same emphasis as the demonstration of its analyticity. Secondly, Frege frequently appeals to primitive truths and their natural order; by contrast the reconstruction I will propose uses only the notion of a basic law of logic or of arithmetic, and it uses both notions in a philosophically neutral sense. In particular the reconstruction does not assume that basic laws reflect a "natural order of primitive truths"; however, it does follow *Grundlagen* in *not* explaining *neither needing nor admitting of proof* in terms of self-evidence. There has been some confusion about the role of self-evidence in *Grundlagen* which it is important to clear-up.[10]

In *Grundlagen* (sect. 5) Frege considers the Kantian thesis that facts about particular numbers are grounded in intuition. His criticism of this thesis is subtle and hinges on a dialectical use of self-evidence in the premise that when a truth fails to qualify as self-evident, our knowledge of it cannot be intuitive. Frege argues that since facts about very large numbers are not self-evident, our knowledge of such facts cannot be intuitive and, hence, Kantian intuition cannot account for our knowledge of *every* fact about particular numbers. Here it is important to recognize that Frege is not proposing that the justification of primitive truths *must* rest on their self-evidence. Such a "justification" would be incompatible with Frege's rejection of psychologism. Rather than asserting that primitive truths are justified by their self-evidence, Frege is pointing to an internal tension in what he understands to be the Kantian view of the role of self-evidence in justification and its relation to intuitive knowledge.

Frege goes on to argue – independently of the considerations we have just reviewed – that even if facts about particular numbers were intuitive, and therefore self-evident, they would be unsuitable as a basis for our knowledge of arithmetic. The difficulty is that facts about particular numbers are infinitely numerous. Hence to take them as *the* primitive truths or first principles would "conflict with one of the requirements of reason, which must be able to embrace all first principles in a survey" (*Gl*, sect. 5). So whether or not facts about particular numbers are intuitively given and self-evident, they cannot exhaust the principles on which our arithmetical knowledge is based, and Frege concludes that the Kantian notion of intuition is at the very least not a complete guide to arithmetic's primitive truths.

The positive argument of *Grundlagen* divides into two parts. The first, and by far the more intricate argument, addresses the problem of apriority. The second argument, which I will ignore except in so far as it illuminates the argument for apriority, is directed at showing the basic laws of arithmetic to be analytic.

A principal premise of the argument for apriority – a premise which is established in *Grundlagen* – is Frege's theorem, i.e. the theorem that PA^2 is recoverable as a definitional extension of the second-order theory – called

[10] In this connection, see, for example, Jeshion (2001).

FA, for *Frege arithmetic* – whose sole nonlogical axiom is a formalization of the statement known in the recent secondary literature as *Hume's principle*:

For any concepts *F* and *G*, the number of *F*s is the same as the number of *G*s if, and only if, there is a one-to-one correspondence between the *F*s and the *G*s.

Frege introduces Hume's principle in the context of a discussion (*Gl*, sects. 62–3) of the necessity of providing a *criterion of identity* for number – an informative statement of the condition under which we should judge that the same number has been presented to us in two different ways, as the number of two different concepts. Hume's principle is advanced as such a criterion of identity – a criterion by which to assess such *recognition judgments* as:

The number of *F*s is the same as the number of *G*s.

It is essential to the understanding of Frege's proposed criterion of identity that the cardinality operator – 'the number of (. . .)' – be interpreted by a mapping from concepts to objects. But Hume's principle may otherwise be understood in a variety of ways: as a partial contextual definition of the concept of number, or of the cardinality operator; or we could follow a suggestion of Ricketts (1997, p. 92) and regard Hume's principle not as a definition of number – not even a contextual one – but a definition of the second-level relation of *equinumerosity* which holds of first-level concepts.[11] However we regard it, the criterion of identity possesses the generality that is required of the premise of an argument that seeks to establish the apriority of a known truth. But basing arithmetic on Hume's principle achieves more than its mere derivation from a principle that does not mention a particular object: it effects an account of the basic laws of *pure* arithmetic that reveals their basis in the principle which controls the *applications* we make of the numbers in our cardinality judgments. Since Hume's principle is an arithmetical rather than a logical principle, the derivation of PA^2 from FA is not the reductive analysis that logicism promised. Nonetheless, it is of considerable epistemological interest since, in addition to recovering PA^2 from a general law, the derivation of PA^2 from FA is based on an account of number which explains the peculiar generality that attaches to arithmetic: *in so far as the cardinality operator acts on concepts, arithmetic is represented as being as general in its scope and application as conceptual thought itself.*

The criterion of identity is the cornerstone of Frege's entire philosophy of arithmetic. The elegance of his account of the theory of the natural numbers is founded on his derivation of the Dedekind infinity of the natural

[11] I am assuming that all these construals acknowledge the existence and uniqueness assumptions which are implicit in the use of the cardinality operator as well as the operator's intended interpretation as a mapping from concepts to objects.

numbers from the second-order theory whose sole axiom is the criterion of identity. In *Principia Mathematica*, Whitehead and Russell (1910 and 1912) proceeded very differently. Contrary to conventional opinion, the difficulty with Whitehead and Russell's postulation of an "Axiom of Infinity" is not that it simply assumes what they set out to prove. For this it certainly does not do. *Principia*'s Axiom of Infinity asserts the existence of what Russell called a *noninductive* class of individuals: a class, the number of whose elements is not given by any *inductive* number, where the inductive numbers consist of zero and its progeny with respect to the relation of immediate successor. A *reflexive class* is Russell's term for a Dedekind-infinite class. Russell was well aware of the fact that the proof that every noninductive class is reflexive depends on the Axiom of Choice. Russell did not simply postulate the Dedekind infinity of *Principia*'s reconstructed numbers, but derived it as a nontrivial theorem from the assumption that the class of individuals is noninductive.[12] The difficulty with *Principia*'s use of the Axiom of Infinity is therefore not that it assumes what it sets out to prove; the difficulty is that the axiom is without any justification other than the fact – assuming it is a fact – that it happens to be true. As we will see, this undermines *Principia*'s account of arithmetic by leaving it open to the objection that even if its reconstruction of the numbers is based on true premises, it cannot provide a correct analysis of the epistemological basis for our belief that the numbers are Dedekind-infinite.

Frege avoided an appeal to a postulate like *Principia*'s by exploiting a subtlety in the logical form of the cardinality operator to argue that if our conception of a number is of something that can fall under a concept of first level, then the criterion of identity ensures that the numbers must form a Dedekind-infinite class. The idea that underlies Frege's derivation of the infinity of the numbers from his criterion of identity is the assumption that the cardinality operator is represented by a mapping from concepts to objects. In Frege's hierarchy of concepts and objects, the value of the mapping for a concept as argument has a logical type that allows it to fall under a concept of first level. This has the consequence that any model of Hume's principle must contain infinitely many objects. Beginning with the concept under which nothing falls, and whose number is by definition equal to zero, Frege is able to formulate the notion of a series of concepts of strictly increasing cardinality

[12] *Principia*, vol. II, *124.57. For a discussion of the relation between the notion of infinity assumed by *Principia*'s axiom and Dedekind infinity, as well as a discussion of Russell's proof – without the Axiom of Choice, but assuming the Axiom of Infinity – that the cardinal numbers of *Principia* form a Dedekind-infinite class, see Boolos (1994).

having the property that, with the exception of the first concept, every concept in the series is defined as the concept that holds of the numbers of the concepts which precede it. The idea of such a series of concepts is the essential step in Frege's derivation of PA^2 from FA. For Frege this derivation was a premise in a more general argument that was intended to illuminate the relation between arithmetic and logic. Frege had hoped to complete the argument by appealing to a demonstration of the existence of a domain of "logical objects" – classes or extensions of concepts – that could be identified with the numbers. His inability to establish the existence of such a domain has tended to obscure his achievement relative to later developments.

To be sure, Hume's principle, like *Principia*'s Axiom of Infinity, holds only in infinite domains. But it does not follow that the two accounts stand or fall together. A reconstruction of the numbers on the basis of the Axiom of Infinity is extraneous to arithmetic, both in its content and in its account of the epistemic basis of arithmetic. Frege, however, provided an account of the Dedekind infinity of the numbers in terms of their criterion of identity and the logical form of the cardinality operator. A central aspect of his account is lost in a reconstruction like *Principia*'s or one that treats numbers as higher-level concepts rather than objects. For, on any such account, the number of entities of any level above the type of individuals – and therefore also the number of numbers – depends on a parameter – namely, the cardinality of the class of possible arguments to first-level concepts – whose value can be freely set. But then on such a reconstruction the cardinality of the numbers can be freely specified.

By contrast, on Frege's account we can certainly consider domains of only finitely many arguments to first-level concepts – including domains consisting of a selection of only finitely many numbers. But a domain contains the numbers only if it satisfies their criterion of identity, and this forces the domain to be infinite, and the numbers Dedekind-infinite. Hence, whatever the technical interest of a reconstruction of the numbers like *Principia*'s or one that reconstructs the numbers as higher-level concepts, it does not share what is arguably the philosophically most interesting feature of Frege's theory of number: Frege's theory preserves the epistemological status of the pure theory of number by showing that the infinity of the numbers is a consequence of his analysis of number in terms of the criterion of identity.

1.4 ANALYSIS VERSUS JUSTIFICATION

A subtlety in the logical form of Hume's principle that we have emphasized makes it all the more compelling that the account of neither needing nor admitting of proof should not rest on a naive conception of self-evidence. The strength of Hume's principle derives from the fact that the cardinality operator is neither type-raising nor type-preserving, but maps a concept of whatever "level" to an *object*, which is to say, to a possible argument to a concept of *lowest* level. Were the operator not type-lowering in this sense, Frege's argument for the Dedekind infinity of the natural numbers would collapse. The fact that only the type-lowering form of the cardinality operator yields the correct principle argues against taking neither needing nor admitting of proof to be captured by self-evidence: it might be that only one of the weaker forms of the principle drives the conviction of "obviousness," "undeniability," or "virtual analyticity" that underlies what I am calling naive conceptions of self-evidence. Although it is highly plausible that the notion of equinumerosity implicit in our cardinality judgments is properly captured by the notion of one-to-one correspondence, a further investigation is needed to show that Hume's principle is "self-warranting," or whatever one takes to be the appropriate mark of neither needing nor admitting of proof.

Frege was at times highly sympathetic to the idea that fundamental principles can be justified on the basis of their sense. In *Function and Concept*[13] he endorsed the methodology of arguing from the grasp of the sense of a basic law to the recognition of its truth in connection with Basic Law v:

For any concepts *F* and *G*, the extension of the *F*s is the same as the extension of the *G*s if, and only if, all *F*s are *G*s and all *G*s are *F*s.

This lends plausibility to the relevant interpretative claim, but the fact that Law v arguably *does* capture the notion of a Fregean extension poses insurmountable difficulties in the way of accepting this methodology as part of a credible justification of it. Being analytic of the notion of a Fregean extension does not show Basic Law v to be analytic, true, or even consistent. If therefore it is a mark of primitive truths that our grasp of them suffices for the recognition of their truth, then some at least of Frege's basic laws are not primitive truths. Although grasping the sense of a basic law does not always suffice for the recognition of its truth, Frege never appears to have had a more considered methodology for showing that we are justified in believing his basic laws. He seems to have taken for granted that the basic laws of logic and arithmetic are self-warranting and that this is an assumption to which all parties to the discussion are entitled.

I have been concerned to show that Frege's emphasis on the generality of the premises employed in a proof of apriority – rather than their

[13] Frege (1891, p. 11 of the original publication).

independence from experience – is sustainable independently of the truth of logicism. Although Frege's analysis would preserve the notion that the principles of arithmetic express a body of truths that are known independently of experience, there is a respect in which it is independent even of the weaker claim that Hume's principle is a known truth. For, even if the traditional conception of arithmetic as a body of known truths were to be rejected, it would still be possible to argue that Hume's principle expresses the condition on which our application of the numbers rests. As such, it captures that feature that makes the numbers "necessities of thought" and gives our conception of them the constitutive role it occupies in our conceptual framework.

If we set to one side the question of the truth of Hume's principle, there is a fact on which to base the claim that it is foundationally secure, albeit in a weaker sense than is demanded by either the traditional or the Fregean notion of apriority. By the converse to Frege's theorem FA is recoverable from a definitional extension of PA^2, or equivalently, FA is interpretable in PA^2. As a consequence, FA is *consistent relative to* PA^2; a contradiction is derivable in FA only if it is derivable in PA^2, a fact that follows from Peter Geach's observation[14] that when the cardinality operator is understood as a type-lowering mapping from concepts to objects, the ordinals in $\omega + 1$ form the domain of a model of FA. Since Frege regards the basic laws of arithmetic to be known truths, the interpretability of FA in PA^2 would certainly count as showing that FA is foundationally secure as well. But Frege would likely have regarded such an argument as superfluous since Hume's principle was for him also a known truth, and therefore certainly foundationally secure. This shows why an appeal to the *consistency strength* of FA in support of Frege's foundational program must be judged altogether differently from its use in the program which motivated the concept's introduction into the foundations of mathematics.

The study of the consistency strength of subtheories of PA^2 is an essential component of the program we associate with Hilbert, namely, to establish the consistency of higher mathematics within a suitably restricted "intuitive" or "finitistic" mathematical theory. Gödel's discovery of the unprovability of the consistency of first-order Peano arithmetic (PA) within PA (by representing the proof of the incompleteness of PA within PA) motivated the investigation of subtheories of PA, incapable of proving their own consistency, and extensions of PA, capable of proving the consistency of PA. The theory Q known as Robinson arithmetic is a particularly simple

[14] In Geach (1976). See Boolos (1987) for a full discussion.

example of a theory incapable of proving its own consistency, and it forms the base of a hierarchy of increasingly stronger arithmetical theories.[15] But the study of this hierarchy is not integral to the foundational program of Frege for whom knowledge of the truth, and therefore the consistency of PA^2 (and thus of PA), is simply taken for granted.

Of the two foundational programs, only Hilbert's holds out any promise of providing a foundation which carries with it any real justificatory force. Frege's foundational focus differs from Hilbert's in precisely this respect. The goals of Frege's logicism are epistemological, but they are not those of making the basic laws of arithmetic more secure by displaying their basis in Hume's principle, or indeed in logic. To use our earlier terminology, Frege's derivation of PA^2 from FA is part of an *analysis* of arithmetical knowledge rather than a *demonstration* of it.[16]

Indeed, Hilbert's proposal for securing PA^2 and its set-theoretical extensions on an intuitive basis runs directly counter to Frege's goal of showing arithmetic to be analytic and hence independent of intuition. This is because Hilbert's view depends on taking as primitive the idea of iteration (primitive recursion) and our intuition of sequences of symbols:

> [T]hat the [symbolic objects] occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the object, as something that neither is reducible to something else nor requires such reduction. (Hilbert 1925, p. 376)

An attractive feature of the program of basing the consistency of PA^2 on some particularly weak fragment of arithmetic is that it held out the promise of explaining *why* the theory on which it proposed to base the consistency of

[15] As Saul Kripke has observed, since in Q one cannot even prove that $x \neq x + 1$, a natural model for Q is the cardinal numbers with the successor of a cardinal x defined as $x + 1$. Kripke's observation is reported in Burgess (2005, p. 56). Burgess's book describes where a variety of subtheories and extensions of FA are situated in relation to this hierarchy; it should be consulted by anyone interested in the current state of art for such results.

[16] See John Bell (1999b) for a formulation of the relationship between certain distinguished models of Hume's principle – "Frege structures" – and models of the standard set-theoretic reconstruction of Peano–Dedekind arithmetic ("Peano–Dedekind models"). Bell proves the existence of a procedure f which converts a minimal Frege structure into a Peano–Dedekind model and a procedure g which converts a Peano–Dedekind model into a minimal Frege structure; he then shows that f and g are mutually inverse to one another. This has as a consequence the equivalence of the existence of Peano–Dedekind models and the existence of minimal Frege structures. The importance of this fact is not foundationalist in the sense that it shows one or another conception to be a more secure basis for our knowledge of number; such an interpretation is evidently excluded by the interchangeability of the two representations. Rather, the fact that f and g are mutually inverse to one another shows that the notions of a Peano–Dedekind model and a minimal Frege structure provide two different but thoroughly interchangeable representations of our informal conception of number, one associated with the pure theory of number, the other with its application in judgments of cardinality.

PA^2 is plausibly viewed as self-warranting. The notions which such a theory permits us to express – the intuitability of finite sequences of symbols and the iterability of basic operations involving them – are arguably presupposed in the formulation of *any* theory. But a theory that rests on no more and no less than what is demanded by the formulation of a mathematical theory is clearly as warranted as any such theory.

Although Frege's *Grundgesetze* contains a theory of general principles by which to explain the apriority of our knowledge of arithmetic, it requires an appeal to arithmetic's basis in logic. Rather surprisingly – and independently of the issue of consistency – the nature and intended scope of the system of logic of *Grundgesetze* prevent it from possessing the simple intuitive appeal of Hilbert's proposal. Were it not for Gödel's second incompleteness theorem, Hilbert's approach to the consistency of PA^2 would have constituted a successful transcendental justification of it, since it would have shown that PA^2 is consistent relative to the framework of assumptions that underlie the possibility of formulating a mathematical theory. Hilbert therefore promised to carry the argument for the apriority of arithmetic in a direction that is genuinely minimalist in its use of primitive assumptions. This cannot be said of the system of *Grundgesetze*. For this reason, Hilbert's account is more recognizably a traditional foundation or justification for arithmetic – indeed, for mathematics generally – than Frege's.

Our emphasis has been on Frege's analysis of those basic laws of arithmetic which ensure the Dedekind infinity of the numbers. Early on, Hilbert expressed concern with the method by which the existence of a Dedekind-infinite concept is proved in the logicist tradition.[17] But the principal divergence between logicist and nonlogicist approaches to arithmetic arises in connection with the explanation of the remaining law – the principle of mathematical induction. Frege explains the validity of reasoning by induction by deriving the principle from the definition of the natural numbers as the class of all objects having all hereditary properties of zero. Anti-logicists explicitly reject this explanation, since it rests on the questionable idea of the totality of all properties of numbers; they argue that in light of the paradoxes our confidence in this notion is not justified.

Although it was conceived in ignorance of the set-theoretic paradoxes and the analysis of their possible source, there is a sense in which the program of recovering arithmetic from FA retains its interest and integrity even in light of the paradoxes: Frege's definition of the natural numbers shows that there is also a *basis* – if not an entirely satisfactory *justification* – for

[17] See especially the paragraphs devoted to Frege and Dedekind in Hilbert (1904, pp. 130–1).

characteristically arithmetical modes of reasoning in general reasoning, and therefore in conceptual thought itself. It is a further discovery that pursuing arithmetic from this perspective is susceptible to a doubt that might appear avoidable if we take as primitive the idea of indefinite iteration. But it remains the case that there is a compelling account of reasoning by induction that recovers it from general reasoning about concepts, even if our confidence in such an account is diminished by various analyses of the paradoxes.[18]

1.5 SUMMARY AND CONCLUSION

Frege extracted from his reflections on Kant several desiderata that an account of number should meet for it to be independent of intuition. First, such an account must explain the *applicability* of numerical concepts without invoking any intuitions beyond those that are demanded by the concepts to which the numerical concepts are applied. Secondly, it must explain our understanding of recognition judgments without recourse to intuition, and in terms of relations among concepts. Thirdly, the account must recover any piece of arithmetical reasoning that rests on the possibility of indefinitely iterating an operation as a species of general reasoning. Frege tells us (*Gg*, p. ix) that the "fundamental thought" on which his analysis of the natural or counting numbers is based is the observation (formulated in sect. 46 of *Grundlagen*) that a statement of number involves the predication of a concept of another concept; numerical concepts are concepts of second level, which is to say, concepts under which concepts of first level – concepts which hold of objects – are said to fall. This yields an analysis of the notion of a numerical property, as when we predicate of the concept *horse which draws the King's carriage* the property of having four objects falling under it.

To pass from the analysis of numerical properties to the numbers, Frege introduced the cardinality mapping – *the number of Fs*, for *F* a sortal concept – which is "implicitly defined" by the *criterion of identity*:

The number of *F*s is the same as the number of *G*s if and only if the *F*s and the *G*s are in one-to-one correspondence.

This criterion is the basic principle upon which Frege's development of the theory of the natural numbers is based: it states the condition under which judgments to the effect that the number of *F*s is the same as the number of *G*s are true.

18 Particularly Russell's; see Chapter 10 below for a discussion of his analysis of the paradoxes.

The intuitive idea underlying Frege's derivation of the Dedekind infinity of the numbers is the definition of an appropriate sequence of representative concepts. The existence of a number for each of the concepts in the sequence is established using only logically definable concepts and those objects whose existence and distinctness from one another can be proved on the basis of the criterion of identity. Nowhere in this construction is it necessary to appeal to extensions of concepts. In its mathematical development Frege's analysis thus provides an account of our understanding of the finite cardinal numbers and the first infinite cardinal, one which can be carried out independently of the portion of his system which led to inconsistency, and one which is demonstrably consistent.

Frege's earliest contribution to the articulation of logicism consisted in showing that the validity of reasoning by induction can be explained on the basis of our knowledge of principles of reasoning that hold in *every* domain of inquiry. This directly engages Frege's understanding of the Kantian tradition in the philosophy of the exact sciences, according to which principles of general reasoning peculiar to our understanding must be supplemented by intuition for an adequate theory of arithmetical knowledge. In *Begriffsschrift* Part III Frege showed how the property of, for example, following the number zero in the sequence of natural numbers after some finite number of relative products of successor might be defined without reference to number or finiteness. The principle of mathematical induction is an almost immediate consequence of this definition.

Frege's explanation of the possibility of arithmetical knowledge relies on higher-order features of his logic. The assessment of Frege's success in addressing the basis for our knowledge of the infinity of the numbers and the justification of induction, by contrast with the assessment of his success in connection with the applicability of arithmetic, has stalled on the question whether second-order logic is really logic. This is unfortunate, since it has deflected attention from Frege's claim to have shown that reasoning by induction and our knowledge of a Dedekind-infinite system depend on principles which, whether they are counted as logical or not, enjoy the generality and universality of conceptual thought; they do not therefore rest on a notion of Kantian intuition of the sort Frege sought to refute. As Frege understood it, such a notion would require intuition of objects, and it is precisely a reliance on the intuition of objects that he avoided. It may be necessary to concede that Frege's demonstrations rest on intuition in a familiar, psychologistic, sense of the term; what has been missed is how little is conceded by such an admission.